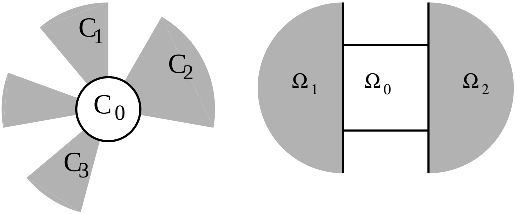
Thermostats, large deviations and Fouriers law

by Errico Presutti, Guido Gentile, Alessandro Giuliani, GG

Thermostat models (Feynman-Vernon 1963): finite system in contact with infinite. Examples



Initial state:

$$\mu_0(dx) \stackrel{def}{=} C\, e^{-\sum_{j=0}^{v} \beta_j H_j(\mathbf{X}_j,\dot{\mathbf{X}}_j)} \prod_j \frac{d\mathbf{X}_j\, d\dot{\mathbf{X}}_j}{N_j!}$$

Equations of motion (thermostat force if $a = 1$)

$$m\ddot{\mathbf{X}}_{0i} = -\,\partial_i U_0(\mathbf{X}_0) - \sum_{j>0} \partial_i U_{0,j}(\mathbf{X}_0, \mathbf{X}_j) + \partial_i \Psi(\mathbf{X}_0) + \Phi_i(\mathbf{X}_0)$$

$$m\ddot{\mathbf{X}}_{ji} = -\,\partial_i U_j(\mathbf{X}_j) - \partial_i U_{0,j}(\mathbf{X}_0, \mathbf{X}_j) + \partial_i \Psi(\mathbf{X}_j) - a\,\alpha_j \dot{\mathbf{X}}_{ji}$$

$U_j(\mathbf{X}_j) = \sum_{q,q' \in \mathbf{X}_j} \varphi(q - q')$, j-th thermostat energy

$U_{0,j}(\mathbf{X}_0, \mathbf{X}_j) = \sum_{q \in \Omega_0, q' \in \Omega_j} \varphi(q - q')$, j-th thermostat-system interac.

$\Psi(X) = \sum_{q \in X} \psi(q)$, Wall potential

Initial state: infinite Gibbs at given density δ_j and temperatures β_j^{-1}

If no phase transitions $\Rightarrow$ kinetic-potential energy density, density *etc* are constant with μ_0-probability 1 at time $t = 0$: examples

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda\cap\Omega_j|} K_{j,\Lambda}(x) = \frac{d}{2}\beta_j^{-1}\delta_j, \qquad \lim_{\Lambda\to\infty} \frac{1}{|\Lambda\cap\Omega_j|} N_{j,\Lambda}(x) = \delta_j$$

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda\cap\Omega_j|} U_{j,\Lambda}(x) = u_j$$

- Thermostats evolution: should be limit of finite volume ???
- Macroscopic thermostats data: (T_j, δ_j, u_j) should be constant ???
- Equivalence of thermostats: $a = 0$ SAME $a = 1$, ????

More formally:

• Regularize: enclose system in ball $\Lambda_n = \Omega \cap \mathcal{B}(R)$ radius $R = 2^n r_\varphi$

• $\Rightarrow$ Time evolutions $x \to S_t^{(n,a)} x$, $a = 0, 1$ have limits as $n \to \infty$??

• **should also be** also $\lim_{n\to\infty} S_t^{(n,a)} x = S_t^{(0)} x \;\; a = 0, 1$ **??**

• Temperature, density, energy density **should be constant** $\forall t, j > 0$, e.g.

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda \cap \Omega_j|} K_{j,\Lambda}(S_t^{(0)} x) = \frac{d}{2} \beta_j^{-1} \delta_j \equiv \frac{d}{2} k_B T_j \delta_j \quad \textbf{??}$$

Entropy: thermostats entropy "increases" by

$$\sigma_0(x) = \sum_{j>0} \frac{Q_j}{k_B T_j(x)}, \qquad Q_j \stackrel{def}{=} -\dot{\mathbf{X}}_j \cdot \partial_{\mathbf{X}_j} U_{0,j}(\mathbf{X}_0, \mathbf{X}_j)$$

History:

Existence: Theorem by Caglioti, Marchioro, Pulvirenti (2000), $d = 3$

Remarkable conclusion of a series of works by

Lanford (1968) 1 dimension (constructive, for "general" states)

Sinai (1971) 1 dimension (a.e. general states, "cluster dynamics")

Marchioro, Pellegrinotti, Presutti (1974) (a.e. only for Gibbs, $\forall d$)

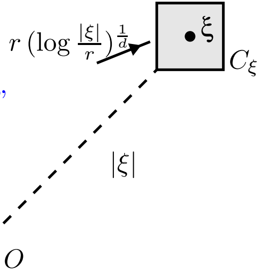
Dobrushin Fritz (1975) (a.e. for dim.=2 general states)

Control via specific energy in large balls: of radius $R \equiv R_n \stackrel{def}{=} 2^n\, r$

$$W(x;\xi,R) \overset{def}{=} \text{total } \frac{\text{energy}}{\varphi_0} + \text{number of particles in ball } \mathcal{B}(\xi,R)$$

$$\mathcal{E}(x) \overset{def}{=} \sup_{\xi} \sup_{R>(\log_+(\frac{\xi}{r_\varphi}))^{1/d}} \frac{W(x;\xi,R)}{R^d}$$

This means:
densities constant on "*log*-scale"



Large deviation for $\mu_0 \Rightarrow \mathcal{E}(x) < +\infty$.

Theorem: $\exists C(\mathcal{E}), c(\mathcal{E})^{-1} \uparrow \mathcal{E}$ *and if* $q_i(0) \in \Lambda_k$ $(v_1 = \sqrt{\frac{2\varphi(0)}{m}})$

$$(1) \qquad |\dot{q}^{(n,0)}(t)| \le v_1\, C(\mathcal{E})\, k^{1/2},$$

$$(2) \qquad \text{distance}\big(q_i^{(n,0)}(t), \partial(\cup_j \Omega_j \cap \Lambda)\big) \ge c(\mathcal{E})\, k^{-3/2\alpha}\, r_\varphi$$

$$(3) \qquad \mathcal{N}_i(t,n) \le C(\mathcal{E})\; k^{3/4}$$

$$(4) \qquad |x_i^{(n,0)}(t) - x_i^{(0)}(t)| \le C(\mathcal{E})\, r_\varphi\, e^{-c(\mathcal{E}) 2^{nd/2}}$$

$\forall\, n > k$. *The* $x^{(0)}(t)$ *is unique frictionless motion satisfying 1,2,3.*

$$(5) \qquad \lim_{n\to\infty} S_t^{(n,1)}(x) \equiv S_t^{(0)}(x), \quad \text{with } \mu_0\text{-probability } 1$$

Λ_n–regularized Gaussian thermostats: thermostats force $\alpha_{j,n}$ so fixed that $U_{j,\Lambda_n} + K_{j,\Lambda_n} = E_{j,\Lambda_n}$ is exact constant of motion

$$\alpha_{j,n} \stackrel{def}{=} \frac{Q_j}{d\,N_j k_B T_j(x)}, \qquad Q_j \stackrel{def}{=} -\dot{\mathbf{X}}_j \cdot \partial_j U_{0,j}(\mathbf{X}_0, \mathbf{X}_j)$$

with $m\dot{\mathbf{X}}_j^2 \stackrel{def}{=} 2K_{j,\Lambda_n}(x) \stackrel{def}{=} d\,N_j k_B T_j(x)$

Idea: Why to expect Equivalence? (in therm. lim. $\Lambda_n \to \infty$)

$$Q_j \stackrel{def}{=} -\dot{\mathbf{X}}_j \cdot \partial_j U_{0,j}(\mathbf{X}_0, \mathbf{X}_j) \;=\; O(1)$$

(Williams,Searles,Evans 2004), **hence**

$$\alpha_{\mathbf{j}} = \frac{\mathbf{Q_j}}{\mathbf{d\,N_j k_B T_{j,n}(x)}} \quad \xrightarrow[n\to\infty]{} \mathbf{0}.$$

But is $T_{j,n}(x) \geq c > 0$ **??** *not* $\forall x!$ $\Rightarrow$ Large deviation

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Theorem (Presutti, G): *with μ_0–probability* 1

(a) $\frac{\mathbf{K_{j,\Lambda_n}(x)}}{|\mathbf{\Lambda_n\cap\Omega_j}|} \;\ge\; \frac{1}{4}\, d\,\delta_j\, k_B T_j$ *(hence $\alpha \xrightarrow[n\to\infty]{} 0$).*

(b) $\lim_{n\to\infty} S_t^{(n,1)}x = \lim_{n\to\infty} S_t^{(n,0)}x$ for all $t>0$.

(c) $\frac{d\mu_t(dx)}{dt} \;=\; -\sigma(x)\,\mu_t(dx)$ and

$$\sigma(\mathbf{x}) = \sum_{\mathbf{j>0}} \frac{\mathbf{Q_j}}{\mathbf{k_B\,T_j(x)}} + \beta_{\mathbf{0}}(\dot{\mathbf{K}}_{\mathbf{0}} + \dot{\mathbf{U}}_{\mathbf{0}} + \dot{\mathbf{\Psi}}_{\mathbf{0}}) \stackrel{\mathbf{def}}{=} \sigma_{\mathbf{0}}(\mathbf{x}) + \dot{\mathbf{F}}(\mathbf{x})$$

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(1) Small kinetic energy is possible but large deviation

(2) Entropy production = volume contraction + a time derivative:

$\Rightarrow$ (average of σ) $\equiv$ (average of σ_0)

provided $\beta_j(x)$ is a constant of motion as $n\to\infty$ and $\beta_j(S_t x)=\beta_j$: very generally phase space contraction = physical entropy production.

Method: "*Entropy estimates*" for thermostatted motions control large deviations

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(I) Proof that kinetic energy per particle (in the Λ_n–regularized motion) stays $> \frac{d}{4}\delta_j\,\beta_j^{-1}$ with μ_0-probability 1 for $t \le \Theta$: i.e stays $\ge \frac{1}{2}$ the equipartition value

(II) Proof that the number of particles and their (kinetic+wall) energy in a unit box grows at most with a power $\gamma \in (\frac{1}{2}, 1)$ of $(\log_+(|\xi|/r_\varphi))^{\frac{1}{2}} \cdot (\log n)^\gamma$

Combining ideas of Sinai, Fritz-Dobrushin, and Marchioro, Pellegrinotti, Presutti, Pulvirenti (1975,1976).

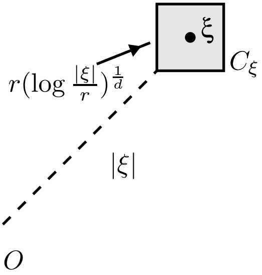
Let ($\log^{\frac{1}{d}}$-growth of energy and energy density with distance to O):

$$||x|| \stackrel{def}{=} \max_{\xi\in\Lambda_n} \frac{\max(N_{C_\xi}(x), \varepsilon_{C_\xi}(x))}{r^d(\log_+(\xi/r_\varphi))^{1/d}}$$

$C_\xi \stackrel{def}{=}$ unit cube centered at ξ, $N_{C_\xi}(x)$= number of particles in C_ξ,

$\varepsilon^2_{C_\xi} \stackrel{def}{=} \max_{q\in C_\xi}(\frac{1}{2}\dot{q}^2 + \psi(q))/\varphi_0$. kinetic + wall energy

densities grow "*log*-distance"



1) Define for x s.t. $\mathcal{E}(x) \leq E$, the **stopping time** $T_n(x)$

$$T_n(x) \stackrel{def}{=} \max\Big\{t : t \leq \Theta : \forall\, \tau < t,$$
$$\frac{K_{j,n}(S_\tau^{(n,1)}x)}{\varphi_0} > \kappa 2^{nd}, \quad \|S_t^{(n,1)}x\|_n < (\log n)^\gamma\Big\}.$$

2) show that before the stopping time frictionless and thermostatted evolution are very close for particles within Λ_k provided cut-off $n \gg k$.

(Indeed within $T_n(x)$, α is very small of order $O(N_n^{-1})$).

Notice also $|\sigma| = O(1)$ depending only on E.

3) Check μ_0-probability of $\mathcal{B} \stackrel{def}{=} \{x \,|\, x \in \mathcal{X}_E \text{ and } T_n(x) \leq \Theta\}$ is

$$\mu_0(\mathcal{B}) \leq C\, e^{-c(\log n)^{2\gamma}}, \qquad \text{Borel-Cantelli.}$$

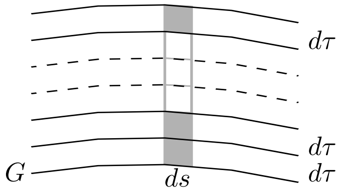
because entropy bounds the μ_0 density of $\mu_0(S_{-t}\cdot)$ before $T_n(x)$ allows us to bound via equilibrium large deviations $\Rightarrow$

Estimate the probability of $\mathcal{X}_n \stackrel{def}{=} \{\mathcal{E}(x) \le E;\, T_n(x) < \Theta\}$.

(2) $\Rightarrow$ bound on the *max entropy production within the stopping time*: $|\int_0^{\tau_n(x)} \sigma(S_t^{(n,1)}x)dt| \le C'$ with C' depending only on E.

For inst. estimate probab. that kinetic energy G becomes $1/2$ of its μ_0-almost sure asympt. value: $G = \frac{1}{4} N_j d\beta_j^{-1}$. IF μ_0 were invariant

$$ds d\tau \stackrel{def}{=} \Big(\int \mu_0(dx)|\dot K|\delta(K-G)\Big)d\tau$$



Remark: *all shaded volumes would have the same μ_0 volume* !

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BBGKY hierarchy, Fourier's equation

(in progress)

by Guido Gentile, Alessandro Giuliani, GG

It is highly desirable to achieve a understanding similar to the one that can be obtained in systems in stationary states out of equilibrium.

In equilibrium systems enclosed in finite containers have a probability distribution with a density on phase space. This is no longer true for systems in steady non equilibrium.

Study existence of stationary states of a hard spheres gas with temperatures at $\pm\infty$ different: $\rho_{\pm\infty}(\mathbf{q}_n)$ correspond to $\rho_\pm$ and $\frac{3}{2}\beta_\pm^{-1} = \langle\, p_i^2 \,\rangle$.

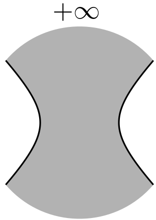


Fig.1: *A hyperboloid-like container* Ω*. Shape is symbolic (d=3)*

Stationary regular BBGKY hierarchy (hard core):

$$\partial_t \rho(\mathbf{p}_n, \mathbf{q}_n; t) = 0 = \sum_{i=1}^{n} \Big(- p_i \cdot \partial_i \rho(\mathbf{p}_n, \mathbf{q}_n; t)$$

$$+ \int_{\sigma(q_i, \mathbf{q}'_n)} \omega \cdot (\pi - p_i)\, \rho(\mathbf{p}_n, \mathbf{q}_n, \pi, q_i + r\omega; t) d\sigma_\omega\, d\pi \Big)$$

$$\stackrel{def}{=} BBGKY_n(\rho(t))$$

$\rho(\mathbf{q}_n, \mathbf{p}_n)$ diff.ble in $|q_i - q_j| > r$ with continuous derivs in $|q_i - q_j| \geq r$.

Has this anything to do with Physics???

- Equation holds (Cercignani) at $t = 0$ in finite volume, smooth initial

$$D_N(\mathbf{p}_N, \mathbf{q}_N) = D_N(\mathbf{p}'_N, \mathbf{q}_N)$$

pair collision continuity (($\mathbf{p}_N, \mathbf{q}_N$) before) and (($\mathbf{p}'_N, \mathbf{q}_N$) after).

$\mathbf{p}'_n = (p_1 \ldots p'_i \ldots p'_j \ldots)$

$\mathbf{p}_n = (p_1 \ldots p_i \ldots p_j \ldots)$

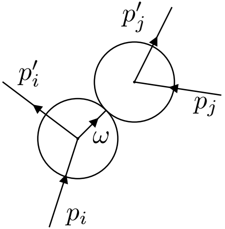


Fig.2

• BUT smoothness lost at $t > 0$ and singularities become dense with t although kept on a set of volume 1 in phase space, and continuity too.

• Nevertheless Lanford has shewn how to derive, by simple iteration,

$$\rho(\mathbf{p}_n, \mathbf{q}_n; t) = \rho(\mathbf{p}_n, \mathbf{q}_n; 0) + \int_0^t BBGKY_n(\rho(t'))\, dt'$$

the B.E. in the Grad limit and Spohn has proved iteration correct in spite of singularities acquired by correlations, densely on phase space

• Thus if a stationary state is studied the equation looks precarious "workig hypothesis".

Heuristic approach: look for solutions of the regular BBGKY equation

Boundary conditions:

(1) **Equilibrium at** $\pm\infty$ at given $T_+ > T_-$:

$$\rho_\emptyset(\mathbf{q}_n) \xrightarrow[\mathbf{q}_n \to \pm\infty]{} \text{equilibrium with suitable activity } z_\pm$$

The condition is not "Gibbs at infinity" but only Gibbsian positional correlations $\Rightarrow$ more freedom but possible interpretation problems.

(2) **Collision continuity**: Inspired from original Maxwell's form of the Boltzmann's equation if suitable factorization of ρ is added **1866**.

(2) **Collision continuity**: if $p_1, p_2 \Rightarrow p'_1, p'_2$ is a collision btwn q_1 and $q_1 + r\omega$ (with $\omega \cdot (p_2 - p_1) < 0$) in the direction ω then (strong form) of continuity (strong) is generally demanded (not Spohn).

$$\rho(\mathbf{q}_n, \mathbf{p}_n) = \rho(\mathbf{q}_n, \mathbf{p}'_n)$$

too strong. It admits a "weak form": for all 1-particle observ. $Q(p)$

$$\sum_{\alpha=1}^{3} \partial_\alpha \int \rho(p,q) p_\alpha Q(p) d^3p =$$

$$\cdot \int_{\omega\cdot(p-\pi)>0} |\omega(\pi - p)| \cdot (Q(p') - Q(p))\, \rho(p, q, \pi, q + r\omega)\, d^3p\, d^3\pi\, d\sigma_\omega$$

and $\forall q \in \Omega$ if $p, \pi \Rightarrow p', \pi'$ after elastic scattering in the cone $d\omega$.

If true $\forall Q$'s equivalent to collision continuity for pair correlations only.

Even this weak continuity might be too strong:

no continuity proof is available: (and it will not be availble for long).

Problems

(1) at volume $< \infty$ and out of eq. correl. not even defined in SRB

(2) if the initial state has the property it keeps it forever (Spohn): however discontinuity might develop at $t = +\infty$

as they do in the other limit of Grad.

Questions:

(a) Are there exact solutions of the BBGKY?

(b) If yes which weak continuity conditon can be imposed?

Answer to (a) **yes**

Answer to (b) $\mathbf{Q(p) = 1, \frac{1}{2}p^2}$ and others, but **not** $\mathbf{Q(p) = p}$!!!

Exact solution:

$$\rho_{even}(\mathbf{q}_n,\mathbf{p}_n)=\rho_\emptyset(\mathbf{q}_n)\prod_{i=1}^n\varphi(q_i,p_i)$$

$$\varepsilon(q)\stackrel{def}{=}\frac{\beta(q)}{\beta_0}-1,\ \varepsilon_0\stackrel{def}{=}\frac{\beta_-}{\beta_+}-1,\ \varphi(q,p)\stackrel{def}{=}\frac{\beta_0}{\beta(q)}\Big(G_{\beta_0}(p)+\varepsilon(q)\delta(p)\Big)$$

Boundary c. imposed at 1-th order in ε_0 (temperature difference) by

$$0=\int_{\omega(p-\widetilde{p})>0}Q(p)dpd\widetilde{p}d\sigma_\omega$$
$$(\rho(q,p,q+r\omega,\widetilde{p})-\rho(q,p',q+r\omega,\widetilde{p}'))\omega\cdot(p-\pi)$$

demanded for $Q(p)=p^2$ for elastic collisions

$$p'=\ p-\omega\cdot(p-\widetilde{p})\,\omega,\qquad \widetilde{p}'=\widetilde{p}+\omega\cdot(p-\widetilde{p})\,\omega$$

and only up to $O((z_0r^3)^{\ell/r},\varepsilon_0^2)$ if $\ell=$ distance of q from $\mathbf{q}_n$ and $\partial\Omega$.

A brief computation shews that this is equivalent to

$$0 = \int_{s(q)\cap\Omega} \rho_{eq}(q, q + r\omega) \left(\beta(q) - \beta(q + r\omega)\right) d\omega$$

At distance ℓ from $\partial\Omega$ the $\rho_{eq}(q, q + r\omega)$ is rotation and translation invariant (up to $O((z_0 r^3)^{\ell/r})$ by K.S. theory of the Mayer expansion.

Hence up to an exp. small error on microscopic scale, $O((z_0 r^3)^{\ell/r})$:

$$\int_{s(q)} \left(\beta(q + r\omega) - \beta(q)\right) d\omega = 0$$

True if $\beta(q)$ is harmonic (*i.e.* if Fourier Law holds).

Also Dirichlet b.c. on $\partial\Omega$ can be considered as well as other geometries can be considered. For instance conic geometry

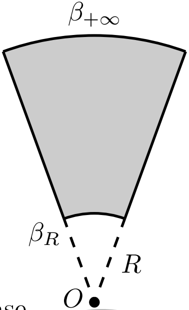


Fig.2: Ω *is a cone with vertex at* O *truncated at a distance* R *from its vertex;* $T(q) = T_0 + \tau(q)$ *solves* $\Delta T = 0$ *with* $\partial_n T = 0$ *on* $\partial\Omega$ *and value* τ_- *at bottom of* Ω *and* $\tau_+ = 0$ *at* ∞*: i.e.* $\tau_- = \frac{\delta}{R}$, $\tau_+ = 0$

special case

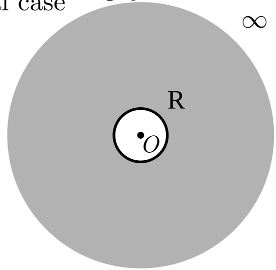


Fig.3: *A special case of Fig.2 the "exterior problem", i.e. the heat conduction outside a ball: a "hot potato" problem. It has an exact solution* $T(q)$.

A geometry with a long cylinder which opens up in two reservoirs

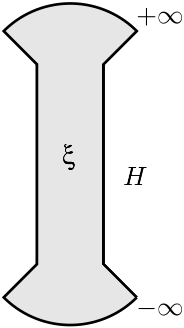


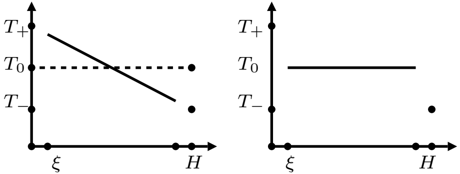
Fig. 4
The container Ω is a cylinder of diameter ξ and height $Hgg\xi \gg r$ continued into two cones extending to ∞.

The interpolating inverse temperature $\beta(q)$ will be close to β_+ at the upper end of the cylinder and close to β_- at the bottom.

An essentially 1-dimensional geometry; temperature values at the top and the bottom (dictated by the b.c. at $\pm\infty$ via the heat equation) will be interpolated essentially linearly ("Saint-Venant's principle"), but $\delta T = O(H^{-1})$.

Very different for Dirichlet ($\beta(q) - \beta_0 = const$ on $\partial\Omega$) and Neumann b.c. ($\partial_n \beta = 0$ on $\partial\Omega$)

Consider both Neumann b.c. ($\partial_n \beta = 0$ on $\partial\Omega$) and Dirichlet ($\beta(q) - \beta_0 = const$ on $\partial\Omega$)



respectively.

The transients at the extremes decay exponentially on scale ξ of the cylinder diameter.

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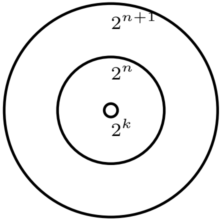
Some Details

Convergence $x_i^{(n,0)}(t) \to \overline{x}_i^{(0)}(t)$, $\qquad q_i(0) \in \Lambda_k$

$$|u_k^n(t) = \max_{q_i(0)\in\Lambda_k} |q_i^{(n,0)}(t) - q_i^{(n+1,0)}(t)|$$



$q_i^{(n,0)}(t) = q_i^{(n,0)}(0) + \dot{q}_i^{(n,0)}(0)\, t + \int_0^t f_i(x^{(n,0)}(\tau)d\tau \Rightarrow$ comparison

Subtract: n and $n+1$ relations $(\eta = \frac{3}{2} + \frac{3}{\alpha}) \;\Rightarrow$

$$u_k^n(t) \leq Cn^\eta \int_0^t u_{k_1}^n(\tau)d\tau \qquad\qquad k_1 = k + C\sqrt{n}$$

#iteration steps $\gg \ell = 2^{n/2} \;\Rightarrow\; |u_k^n(t)| \leq C\frac{(n^\eta\Theta)^\ell}{\ell!}$

Why not "same" for thermostatted dynamics ?

$$u_k^n(t) \le C n^\eta \int_0^\Theta u_{k_1}^n(\tau) d\tau + C 2^{-nd} \qquad k_1 = k + C\sqrt{n}$$

#iteration steps is same $\gg \ell = 2^{n/2}$ BUT

error $\quad C e^{C n^\eta \Theta} 2^{-nd} \to \infty$

Up to Stopping time properties

$$|\dot{q}_i^{(n,1)}(t)| \le C\, v_1 \left(k \, \log n\right)^\gamma, \qquad |q_i^{(n,1)}(t)| \le r_\varphi \left(2^k + C \left(k \, \log n\right)^\gamma\right)$$

$$\Rightarrow \mathcal{N} \le C \left(k \log n\right)^{d\gamma}, \;\; \rho \ge \; c \left(k \log n\right)^{-2(d\gamma+1)/\alpha}$$

Only $(k \log n)^\eta$ particles interact with $q_i \in \Lambda_k$

Compare $x^{(n,1)}(t)$ and $x^{(n,0)}(t)$ ℓ times $2^{k_\ell} = 2^k + \ell\, C\, (k \log n)^\gamma$

Compare $x^{(n,1)}(t)$ and $x^{(n,0)}(t)$ ℓ times $2^{k_\ell} = 2^k + \ell\, C\,(k \log n)^\gamma$ with $\ell \sim 2^n/(\log n)^\gamma$

$$\frac{u_{k_\ell}(t,n)}{r_\varphi} \leq\ C\,(k \log n)^\eta \left(2^{-nd} + \int_0^t \frac{u_{k_{\ell+1}}(s,n)ds}{r_\varphi \Theta}\right)$$

This time the Lyapunov exponent is small

$$\frac{u_k(t,n)}{r_\varphi} \leq e^{C\,(k\ \log n)^\eta} C(k \log n)^\eta 2^{-dn}$$
$$+ \frac{(C\,(k\ \log n)^\eta)^{\ell^*}}{\ell^*!}\, C\,(2^k + k(\log n)^\gamma + k^{1/2})$$