Chaotic hypothesis applicability: quantum or strongly dissipative systems

GG arXiv:cond-mat/0701124
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Non Equilibrium: lack of a model playing role of Ising in 2D.

Microscopic theory of heat conduction: open

Ruelle's: NE Statistics = statistics of almost all initial data chosen with the Liouville's distr.

Non equilibrium $\to$ $m\vec{a} = -\vec{\partial}V(\vec{q}) + \vec{F} + \alpha(\vec{p},\vec{q})\vec{p}$.

Liouville invalid; phase space contracts (in average): its rate has the interpretation of entropy production $\sigma(\vec{q}(n),\vec{p}(n))$ rate

Are there any exact results at all?

If the system is "very chaotic" yes: Fluctuation theorem for stationary state. If $s \stackrel{def}{=} \frac{1}{T}\sum_0^T \frac{\sigma(\vec{q}(n),\vec{p}(n))}{\langle\sigma\rangle}$

$$\frac{Prob(s\in(x,x+\varepsilon))}{Prob(s\in(-x-\varepsilon,-x))} \stackrel{T\to\infty}{=} e^{-T\,x\,\langle\sigma\rangle}$$

$$\frac{Prob(s \in (x, x+\varepsilon))}{Prob(s \in (-x-\varepsilon, -x))} \stackrel{T\to\infty}{=} e^{-T\,x\,\langle\sigma\rangle}$$

No free parameters. Applications?

Cohen-G: Chaotic hypothesis: If a system is chaotic then it can be supposed to be so "as much as possible", *i.e.* it is "Anosov"

This allows to make use of the FT and to "test it".
Of course you do not "test" theorems!

Rather tests chaotic hyp. which has the role of ergodic hyp. in nonequilibrium: more proper to call Fluctuation relation.

(a) many small systems must be quantum: CH? FR? FT?
(b) strong dissipation $\Rightarrow$ "small attractor" is CH reasonable?

Quantum systems

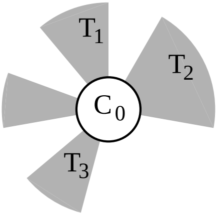


fig.1

Particles in C_0 ("system") interact with particles in shaded regions ("thermostats") constrained to fixed total K.E.

H operator on $L_2(C_0^{3N_0})$, (symm./antisymm.) wave funct.s Ψ,

$$H = -\frac{\hbar^2}{2}\Delta_{\vec{X}_0} + U_0(\vec{X}_0) + \sum_{j>0}\left(U_{0j}(\vec{X}_0, \vec{X}_j) + U_j(\vec{X}_j) + K_j\right)$$

Spectrum consists of eigenvalues $E_n = E_n(\{\vec{X}_j\}_{j>0})$, for $\vec{X}_j$ fixed.

Dynamical sys. on $(\Psi, (\{\vec{X}_j\}, \{\dot{\vec{X}}_j\})_{j>0})$ "phase space" def:

$$-i\hbar\dot{\Psi}(\vec{X}_0) = (H\Psi)(\vec{X}_0), \quad \text{and for } j > 0$$
$$\ddot{\vec{X}}_j = -\Big(\partial_j U_j(\vec{X}_j) + \langle\, \partial_j U_j(\vec{X}_0, \vec{X}_j)\,\rangle_\Psi\Big) - \alpha_j \dot{\vec{X}}_j$$

$$\alpha_j \stackrel{def}{=} \frac{\langle\, W_j\,\rangle_\Psi - \dot{U}_j}{2K_j}, \qquad W_j \stackrel{def}{=} -\dot{\vec{X}}_j \cdot \vec{\partial}_j U_{0j}(\vec{X}_0, \vec{X}_j)$$

$(\langle\cdot\rangle_\Psi \equiv \langle\Psi|\cdot|\Psi\rangle)$. Evolution keeps $K_j \equiv \frac{1}{2}\dot{\vec{X}}_j^2$ exactly constant (defining therm. temp. T_j via $K_j = \frac{3}{2}k_B T_j N_j$, as classical case).

NOT a time dep. Schrödinger eq.: *essential interaction syst-thermos*. This is a classical Dynamical system.

Divergence: $\sigma(x) = \sum_j \frac{Q_j}{k_B T_j} + \frac{\dot{U}_1}{k_B T_1}$ (*same as classical*)

Equations are reversible and chaotic $\Rightarrow$ CH $\Rightarrow$ SRB + FT

Consistency check : *system interacting with a single thermostat* the SRB distribution should be equivalent to the canonical distribution. *True in classical case*).

Candidate for μ: probability proportional to $d\Psi\, d\vec{X}_1\, d\dot{\vec{X}}_1$ times

$$\sum_{n=1}^{\infty} e^{-\beta E_n(\vec{X}_1)} \delta(\Psi - \Psi_n(\vec{X}_1)\, e^{i\varphi_n})\, d\varphi_n\, \delta(\dot{\vec{X}}_1^2 - 2K_1)$$

$\Rightarrow$ (??) expectation of O is a Gibbs state of therm. equil. with a special kind (random $\vec{X}_1, \dot{\vec{X}}_1$) of b.c. and temperature T_1.

$$\langle\, O\,\rangle_\mu = Z^{-1} \int \sum_{n=1}^{\infty} e^{-\beta E_n(\vec{X}_1)} \langle \Psi_n(\vec{X}_1)|O|\Psi_n(\vec{X}_1)\rangle \delta(\dot{\vec{X}}_1^2 - 2K_1) d\vec{X}_1 \dot{\vec{X}}_1$$

But not invariant under evolution: difficult to exhibit explicitly an invariant distribution (why should it be easy? *Aesopus*)

Nevertheless if *adiabatic approximation* (i.e' the classical motion of the thermostat particles is on a time scale much slower than the quantum evolution of the system).

Eigenstates at time 0 evolve following the variations of Hamiltonian $H(\vec{X}(t))$ due to thermostats particles motion, without changing quantum numbers and can check

μ is stationary if β is chosen such that $\beta = \frac{3N_1}{2K_1} \equiv (k_B T_1)^{-1}$ the distribution $\langle \cdot \rangle_\mu$ is stationary.

Conjecture: true SRB is *also* equivalent to Gibbs at temp. $(k_B\beta)^{-1}$

Under time evolution a time $t > 0$ infinitesimal:

$$\vec{X}_1 \;\rightarrow\; \vec{X}_1 + t\dot{\vec{X}}_1 + O(t^2)$$

$$E_n(\vec{X}_1) \;\rightarrow\; E_n + t\,e_n + O(t^2) \qquad \text{with}$$

$$e_n \overset{def}{=} \langle\, \dot{\vec{X}}_1 \cdot \vec{\partial}_{\vec{X}_1} U_{01} \,\rangle_{\Psi_n} + t\dot{\vec{X}}_1 \cdot \vec{\partial}_{\vec{X}_1} U_1 = -t\,(Q_1 + \dot{U}_1)$$

$$e^{-\beta E_n(\vec{X}_1)} \;\rightarrow\; e^{-\beta t e_n}$$

thermostat phase space contracts by $e^{t\sigma} \equiv e^{t\frac{3N_1 e_n}{2K_1}}$

Thus if β is chosen such that $\beta = \frac{3N_1}{2K_1} \equiv (k_B T_1)^{-1}$ the distribution $\langle \cdot \rangle_\mu$ is stationary.

$\Rightarrow$ possibility of defining the temperature via the FT if Q is measurable or Q if T is measurable (originally suggested by Cugliandolo and Kurchan as a possible application of FT)

Consider another non equilibrium case in which CH does not hold exactly. In non equilibrium dissipation often leads to trivial results.

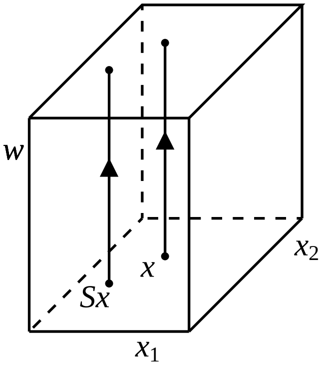
So we ask if at least in a simple case it would be possible to check that the system at least develops a strange attractor.

And can the attractor be described in detail? and proved to be non trivial? *E.g.* not a periodic orbit (either with a period close to an unperturbed one (resonance) or with a period of the external forcing (synchronization)? as it often happens.

Study a system which is “chaotic” but does not satisfy CH assumptions

Natural case: subject a system for which the CH hypotheses would hold to a periodic forcing.

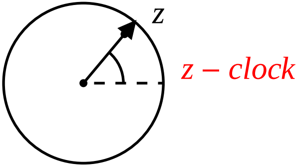
In absence of forcing no quasi periodic motions and few periodic ones: finitely many with period less than any $T < \infty$.



$$Sx = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\dot{x} = \delta_1(z)(Sx - x)$$

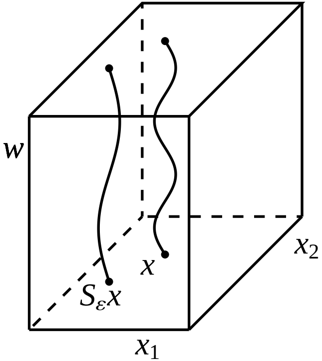
$$\dot{w} = 1 \qquad \dot{z} = 1$$



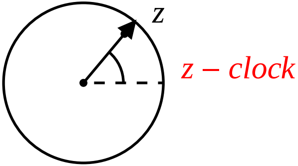
Look at Poincaré's map $\mathcal{S}$ at $t = 2\pi n$

$$x' = Sx, \qquad w' = w$$

planes $w = const$ are invariant. What about perturbations?



$$\dot{x} = \delta_1(z)(Sx - x) + \varepsilon f_\varepsilon(x, w, z)$$
$$\dot{w} = 1 + \varepsilon g_\varepsilon(x, w, z)$$
$$\dot{z} = 1$$

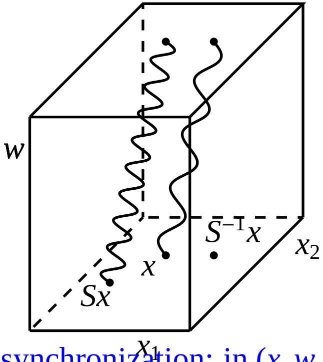


Look at Poincaré's map $\mathcal{S}$ at $t = n$

$$x' = Sx + \varepsilon \overline{f}_\varepsilon(x, w), \qquad w' = w + \varepsilon \overline{g}_\varepsilon(x, w)$$

1gg a) volume preserving (if $\varepsilon = 0$ the $\mathcal{S}$ is not even ergodic, & has one "central" Lyapunov exponent zero.

Theor.: *Near (in C^2) $\bar{f} = \bar{g} = 0$ $\exists$ open set of perturb. with $\mathcal{S}_\varepsilon$*



(1) Ergodic

(2) Central Lyap. exp. $\ell_\varepsilon > 0$

(3) $\exists$ $\mathcal{S}$-invariant foliation Λ into C^1-smooth lines l

(4) $\exists k$ and E of full vol. s.t. $E \cap \ell$ is exactly $k < \infty$ pts (!)

[Conjectures: $k > 1$ & $k = 1$]

SW 1999, RW 2001

No synchronization: in (x, w, z) the planes $w = const$ are invariant under the P.-map but volatilize under perturbation

b) Is synch. possible in dissipation? which attractor structure?

Three possibilities?

1) Simplest possibility attractor=“periodic orbit” .

2) Attractor = “pathological ” (but with Hausdorff dim. < 3).

3) A periodic strange attractor: dissipation stabilizes a single one among the unperturbed invariant surfaces $w = const$

$$\dot{x} = \delta_1(z)(Sx - x) + \varepsilon f_\varepsilon(x, w, z)$$

$$\dot{w} = 1 + \varepsilon g_\varepsilon(x, w, z), \qquad \dot{z} = 1$$

Case $f \equiv 0$

$$\dot{x} = \delta_1(z)(Sx - x)$$
$$\dot{w} = 1 + \varepsilon g_\varepsilon(x, w, z), \qquad \dot{z} = 1$$

Simulations can be consistent with 2) or 3). “Naivest” case

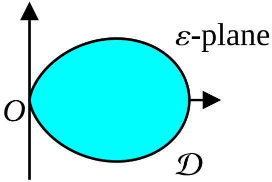
$$f = 0, \qquad g(x, w, z) = (\sin(z - w) + \sin(x_1 + z + w))$$

(first studied) immediately shows an instance of (3). Special:

$$\text{average perturb.} \quad g_0(x, w) \stackrel{def}{=} \int_0^{2\pi} g(x, w + t, t)\frac{dt}{2\pi} \quad = \sin w$$
$$\text{av. compression} \quad g_1(x, w) \stackrel{def}{=} \int_0^{2\pi} \frac{\partial}{\partial w} g(x, w + t, t)\frac{dt}{2\pi} \quad = \cos w$$

Notice that in example $g_0(x, \pi) = 0$, $g_1(x, \pi) = \Gamma < 0$, $\forall x$.

Result: *If* $\exists\, w_0$ *such that* $g_0(x, w_0) = 0,\ g_1(x, w_0) = \Gamma < 0,\ \forall x$,
then $\exists\, \varepsilon_0$ for $0 < \varepsilon < \varepsilon_0$

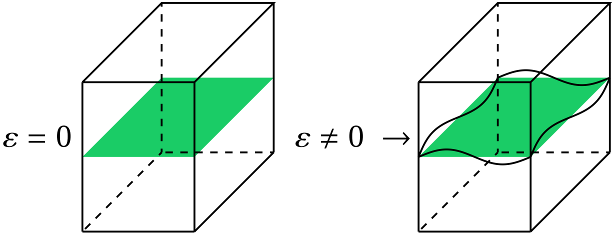


(1) $\exists$ attractor $w(x) = w_0 + U_\varepsilon(x)$

(2) $U_\varepsilon(x)$ is h-Hölder continuous

$$h \geq \frac{|\Gamma|}{\log \lambda_+}\varepsilon + O(\varepsilon^2)$$

(3) U_ε is analytic in ε in $\mathcal{D}$



Conjectures

(a') $\int_0^{2\pi} \mathrm{d}t\, g(x, w_0 + t, t) = \widetilde{g}(x)$, with $\widetilde{g}(x)$ with 0 average,
(b') $\int_0^{2\pi} \mathrm{d}t\, \partial_w g(x, w_0 + t, t) = \widetilde{g}_1(x)$, with $\widetilde{g}_1(x)$ with < 0 average.

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