Resonances and synchronization

by Guido Gentile, Alessandro Giuliani, GG, arXiv:1106.1476

Non Equilibrium: lack of a model playing role of Ising in 2D.

Microscopic theory of heat conduction: open

Ruelle's: NE Statistics = statistics of almost all initial data chosen with the Liouville's distr.

Non equilibrium $\rightarrow$ $m\vec{a} = -\vec{\partial}V(\vec{q}) + \vec{F} + \alpha(\vec{p},\vec{q})\vec{p}$.

Liouville th. invalid; phase space contracts (in average): its rate has the interpretation of entropy production $\sigma(\vec{q}(n),\vec{p}(n))$ rate

Are there any exact results at all?

If the system is "very chaotic" yes: Fluctuation theorem for stationary state. If $s \stackrel{def}{=} \frac{1}{T}\sum_0^T \frac{\sigma(\vec{q}(n),\vec{p}(n))}{\langle\sigma\rangle}$

$$\frac{Prob(s \in (x, x+\varepsilon))}{Prob(s \in (-x-\varepsilon, -x))} \stackrel{T\to\infty}{=} e^{-T\,x\,\langle\sigma\rangle}$$

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No free parameters. Applications?

Cohen-G: Chaotic hypothesis: If a system is chaotic then it can be supposed to be so "as much as possible", *i.e.* it is "Anosov"

This allows to make use of the FT and to "test it".

Of course you do not "test" theorems!

Rather one tests the chaotic hypothesis which takes the role of the ergodic hyothesis in non equilibrium: more properly call Fluctuation relation.

In non equilibrium dissipation is essential and quest for a model that can play the role of Ising remains.

Furthermore dissipation often leads to rather trivial results.

So we ask if at least in a simple case it would be possible to check that the system at least develops a strange attractor.

And can the attractor be described in detail? and proved to be non trivial? *E.g.* not a periodic orbit (either with a period close to an unperturbed one (resonance) or with a period of the external forcing (synchronization)? as it often happens in presence of dissipation.

Periodically forced chaotic systems and chaotic hypothesis

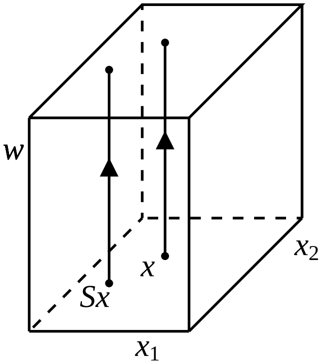
Study a system which is “chaotic” but does not satisfy FT assumptions

A natural case is obtained subjecting a system for which the FT hypotheses would hold to a periodic forcing.

1) volume preserving (“no dissipation”)
2) dissipative

In absence of forcing no quasi periodic motions and few periodic ones: finitely many with period less than any $T < \infty$.

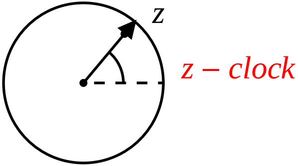
So in this case only “extrinsic” resonances properly can exist: *i.e.* synchronization with externally imposed periodic forces.



$$Sx = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\dot{x} = \delta_1(z)(Sx - x)$$

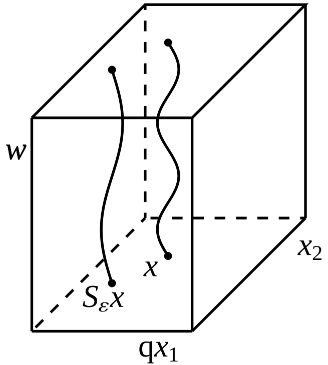
$$\dot{w} = 1 \qquad \dot{z} = 1$$



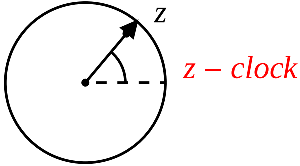
Look at Poincaré's map $\mathcal{S}$ at $t = 2\pi n$

$$x' = Sx, \qquad w' = w$$

planes $w = const$ are invariant. What about perturbations?



$$\dot{x} = \delta_1(z)(Sx - x) + \varepsilon f_\varepsilon(x, w, z)$$
$$\dot{w} = 1 + \varepsilon g_\varepsilon(x, w, z)$$
$$\dot{z} = 1$$

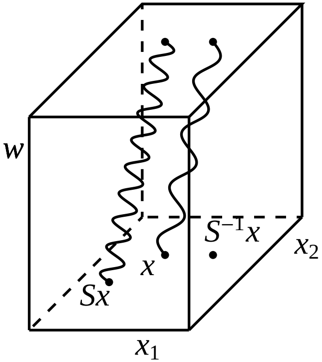


Look at Poincaré's map $\mathcal{S}$ at $t = n$

$$x' = Sx + \varepsilon \overline{f}_\varepsilon(x, w), \qquad w' = w + \varepsilon \overline{g}_\varepsilon(x, w)$$

a) volume preserving (if $\varepsilon = 0$ the $\mathcal{S}$ is not even ergodic, & has one "central" Lyapunov exponent zero.

Theor.: *Near (in C^2) $\bar{f} = \bar{g} = 0$ $\exists$ open set of perturb. with $\mathcal{S}_\varepsilon$*



(1) Ergodic

(2) Central Lyap. exp. $\ell_\varepsilon > 0$

(3) $\exists$ $\mathcal{S}$-invariant foliation Λ into C^1-smooth lines l

(4) $\exists k$ and E of full vol. s.t. $E \cap \ell$ is exactly $k < \infty$ pts (!)

[Conjectures: $k > 1$ & $k = 1$]

SW 1999, RW 2001

No synchronization: in (x, w, z) the planes $w = const$ are invariant under the P.-map but volatilize under perturbation

b) Is synch. possible in dissipation? which attractor structure?

Three possibilities?

1) Simplest possibility attractor=“periodic orbit” .

2) Attractor = “pathological ” (but with Hausdorff dim. < 3).

3) A periodic strange attractor: dissipation stabilizes a single one among the unperturbed invariant surfaces $w = const$

$$\dot{x} = \delta_1(z)(Sx - x) + \varepsilon f_\varepsilon(x, w, z)$$
$$\dot{w} = 1 + \varepsilon g_\varepsilon(x, w, z), \qquad \dot{z} = 1$$

Case $f \equiv 0$

$$\dot{x} = \delta_1(z)(Sx - x)$$
$$\dot{w} = 1 + \varepsilon g_\varepsilon(x, w, z), \qquad \dot{z} = 1$$

Simulations can be consistent with 2) or 3). “Naivest” case

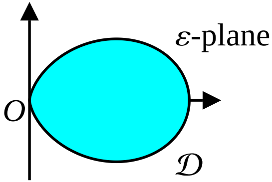
$$f = 0, \qquad g(x, w, z) = (\sin(z - w) + \sin(x_1 + z + w))$$

(first studied) immediately shows an instance of (3). Special:

$$\text{average perturb.} \quad g_0(x, w) \stackrel{def}{=} \int_0^{2\pi} g(x, w + t, t) \frac{dt}{2\pi} \quad = \sin w$$
$$\text{av. compression} \quad g_1(x, w) \stackrel{def}{=} \int_0^{2\pi} \frac{\partial}{\partial w} g(x, w + t, t) \frac{dt}{2\pi} \quad = \cos w$$

Notice that in example $g_0(x, \pi) = 0$, $g_1(x, \pi) = \Gamma < 0$, $\forall x$.

Result: *If* $\exists\, w_0$ *such that* $g_0(x, w_0) = 0,\ g_1(x, w_0) = \Gamma < 0,\ \forall x,$
then $\exists\, \varepsilon_0$ for $0 < \varepsilon < \varepsilon_0$



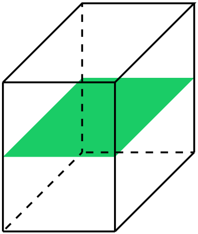
(1) $\exists$ attractor $w(x) = w_0 + U_\varepsilon(x)$

(2) $U_\varepsilon(x)$ is h-Hölder continuous

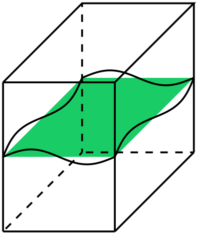
$$h \geq \frac{|\Gamma|}{\log \lambda_+}\varepsilon + O(\varepsilon^2)$$

(3) U_ε is analytic in ε in $\mathcal{D}$

$\varepsilon = 0$



$\varepsilon \neq 0 \quad \rightarrow$



Conjectures

(a') $\int_0^{2\pi} \mathrm{d}t\, g(x, w_0 + t, t) = \widetilde{g}(x)$, with $\widetilde{g}(x)$ with 0 average,
(b') $\int_0^{2\pi} \mathrm{d}t\, \partial_w g(x, w_0 + t, t) = \widetilde{g}_1(x)$, with $\widetilde{g}_1(x)$ with < 0 average.

Basics: write equations and look for $U_\varepsilon(x)$:

$$(x(t), w(t), t) = (x, w + t + u(x,t), t), \qquad t \in (0, 2\pi]$$

$$\begin{aligned} \dot{x} &= \delta(z)(Sx - x) \\ \dot{w} &= 1 + \varepsilon g_\varepsilon(x, w, z) \\ \dot{z} &= 1 \end{aligned} \qquad \begin{aligned} u(x, 0) &= U(x) \\ u(x, 2\pi) &= U(Sx) \end{aligned}$$

Taylor expansion in u to second order (note μ instead of ε)

$$\begin{aligned} \dot{u}(x,t) &= \varepsilon g(x, w + t + u(x,t), t) \\ &\equiv \varepsilon g(x, w_0 + t, t) + \mu\, \partial_w g(x, w_0 + t, t)\, u(x,t) + \varepsilon\, G(x, t, u(x,t)) \end{aligned}$$

Solved "as a linear equation" in terms of "Wronskian"

Power series in ε: check convergence radius for $|\varepsilon| < const\ \sqrt{\mu}$ hence $\mu = \varepsilon$ is possible. Actually convergence for

$$\varepsilon = \rho e^{i\theta}, \qquad if\ \rho < const\,(\cos\theta)^2$$

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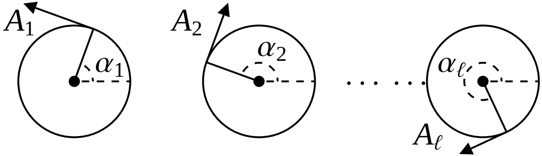
Quasi integrable systems resonances

$$H(\vec{A},\vec{\alpha}) = \frac{1}{2}\vec{A}^2 + ef(\vec{A},\vec{\alpha})$$



Representation of phase space in terms of ℓ rotators:
$\vec{\alpha} = (\alpha_1,\ldots,\alpha_\ell) \in T^\ell$, $\vec{A} = (A_1,\ldots,A_\ell)$
Unperturbed system $\Rightarrow$ have all possible spectra $\vec{\omega}$:

$$\vec{A} = \vec{A}_0, \quad \vec{\alpha} = \vec{\alpha}_0 + \vec{\omega}t, \quad \vec{\omega} = (\omega_1,\ldots,\omega_\ell)$$

in particular ω_j rationally dependent as for $\vec{\omega}^0$:

$$\vec{\omega}^0 = (\omega^0_1,\ldots,\omega^0_{\ell'},0,0,\ldots)$$

$\vec{\omega}^0 = (\omega_1^0, \ldots, \omega_{\ell'}^0, 0, 0, \ldots)$: resonant motions.
More generally "resonances with rotation $\vec{\omega}^0$ are"

$$\begin{cases} \vec{A} = \vec{A}_0 + \vec{X}(\vec{\psi}) \\ \vec{\alpha} = (\vec{\psi} + \vec{Y}(\vec{\psi}), \vec{\alpha}'_0 + \vec{Y}'(\vec{\psi}) \end{cases}, \qquad \vec{\psi} \in T^{\ell'}, \ \ell' < \ell$$

$(\vec{X}, \vec{Y} = \text{smooth})$, the motions $\vec{\psi} \to \vec{\psi} + \vec{\omega}^0 t$ solve eq. of motion.

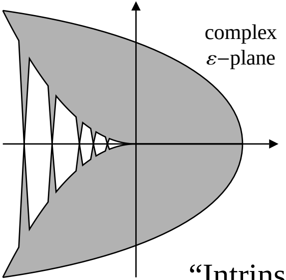
Resonances exist (ε small) under "mild conditions" non trivial

i.e. spectra $(\omega_1^0, \ldots, \omega_{\ell'}^0, 0, 0, \ldots)$ are possible (KAM).

Example $\ell' = 2$: $\vec{\alpha} \stackrel{def}{=} (\alpha_1, a_2, \vec{\alpha}') \in T^2 \times T^{\ell-2}$,

$$f(\vec{\alpha}) \equiv f(\alpha_1, a_2, \vec{\alpha}'), \qquad \overline{f}(\vec{\alpha}') = \int \frac{d\alpha_1 d\alpha_2}{(2\pi)^2} f(\alpha_1, \alpha_2, \vec{\alpha}')$$

$X_\varepsilon(\vec\psi), Y_\varepsilon(\vec\psi)$ analytic in $\varepsilon, \vec\psi$ exist with domain including



$$\overline{f}(\vec\alpha') = \int \frac{d\alpha_1 d\alpha_2}{(2\pi)^2} f(\alpha_1, \alpha_2, \vec\alpha')$$

$$\partial\overline{f}(\vec\alpha\,'^0) = \vec 0,\ \partial^2\overline{f}(\vec\alpha\,'^0) < 0$$

$|\vec\omega^0 \cdot \vec\nu| > C|\vec\nu|^{-\tau}$, "Diophantine"

"Intrinsic" Resonances at real ε (Gentile-G)

$$\begin{cases} \vec A = \vec A_0 + \vec X(\vec\psi) \\ \vec\alpha = (\vec\psi + \vec Y(\vec\psi), \vec a\,'_0 + \vec Y'(\vec\psi)) \end{cases}, \qquad \vec\psi \in T^{\ell'},\ \ell' < \ell$$

Other kinds of resonances are "extrinsic" res. or exhibit synchronization proper:

$$\begin{cases} \dot{\vec{\alpha}} = \vec{A} \\ \dot{\vec{A}} = -\varepsilon \partial_{\vec{\alpha}} V(\vec{\alpha}) + \varepsilon \vec{F}(\vec{\omega} t) \end{cases}$$

$\exists$ motion with spectrum $\vec{\omega}$? $\ell = 2$ (Corsi-Gentile). Friction?