Resonances and synchronization

by Guido Gentile, Alessandro Giuliani, GG, arXiv:1106.1476

1) Quasi integrable systems
2) Chaotic systems

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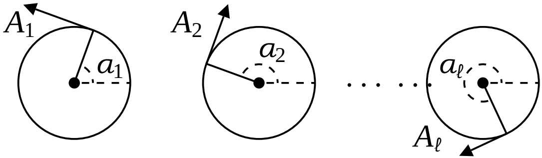
$$H(\vec{A},\vec{\alpha}) = \frac{1}{2}\vec{A}^2 + \varepsilon f(\vec{A},\vec{\alpha})$$



Representation of phase space in terms of ℓ rotators:

$\vec{\alpha} = (\alpha_1,\ldots,\alpha_\ell) \in T^\ell,\ \vec{A} = (A_1,\ldots,A_\ell)$

Unperturbed system ⇒ motions will have all spectra

$$\vec{A} = \vec{A}_0, \quad \vec{\alpha} = \vec{\alpha}_0 + \vec{\omega} t, \quad \vec{\omega} = (\omega_1, \ldots, \omega_\ell)$$

⇒ in particular ω_j rationally depend.: *e.g.* $(\omega_1^0, \ldots, \omega_{\ell'}^0, 0, 0, \ldots)$

Such motions are called resonant: more generally

$$\begin{cases} \vec{A} = \vec{A}_0 + \vec{X}(\vec{\psi}) \\ \vec{\alpha} = R\vec{\psi} + \vec{Y}(\vec{\psi}) \end{cases}, \qquad \vec{\psi} \in T^{\ell'}, \ \ell' < \ell$$

$\vec{X}, \vec{Y}$ = smooth, $R = \ell \times \ell'$ integer matrix and the

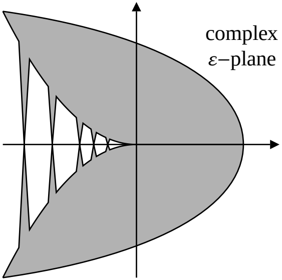
$$\vec{\psi} \to \vec{\psi} + \vec{\omega}^0 t$$

give a solution to the eq. of motion then: resonant.
If $\ell' = 1$ periodic: since Poincaré; $\ell' > 1$

If $1 < \ell' < \ell$ KAM-theory: mild conditions but non trivial

Example $\vec{\alpha} = (\vec{\alpha}_1, \vec{\alpha}_2) \in T^{\ell'} \times T^{\ell-\ell'}$, $\qquad f(\vec{\alpha}) \equiv f(\vec{\alpha}_1, \vec{\alpha}_2)$

$X_\varepsilon(\vec{\psi}), Y_\varepsilon(\vec{\psi})$ analytic in $\varepsilon, \vec{\psi}$ exist with domain including



$\overline{f}(\vec{\alpha}_2) = \int \frac{d\vec{\alpha}_1}{(2\pi)^{\ell'}} f(\vec{\alpha}_1, \vec{\alpha}_2)$

$\partial \overline{f}(\vec{\alpha}_2^0) = \vec{0}, \; \partial^2 \overline{f}(\vec{\alpha}_2^0) < 0$

$|\vec{\omega}^0 \cdot \vec{\nu}| > C|\vec{\nu}|^{-\tau}$, Diophantine p.

Resonances exist at real ε points

(Llave-Zhou, Gentile-G)

"Intrinsic res". "Extrinsic" res. or synchronization:

$$\begin{cases} \dot{\vec{\alpha}} = \vec{A} \\ \dot{\vec{A}} = -\varepsilon \partial_{\vec{\alpha}} V(\vec{\alpha}) + \varepsilon \vec{F}(\vec{\omega} t) \end{cases}$$

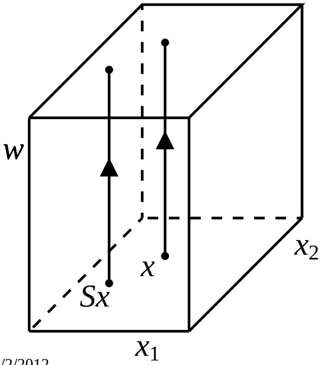
$\exists$ motion with spectrum $\vec{\omega}$? $\ell = 2$ (Corsi-Gentile). Friction?

Chaotic systems: paradigm Anosov flow periodically forced

1) volume preserving

2) dissipative

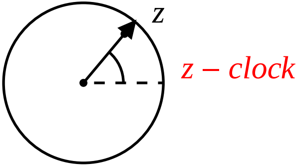
No quasi periodic motions and few periodic ones: finitely many with period less than any $T < \infty$. So in this case only extrinsic resonances properly can exist: synchronization.

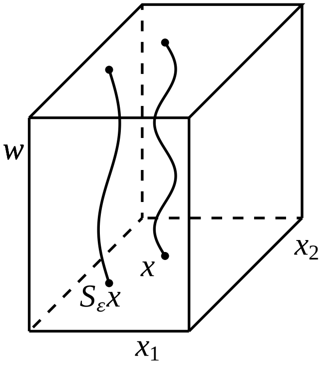


$$Sx = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

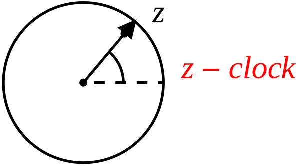
$$\dot{x} = \delta(z)(Sx - x)$$

$$\dot{w} = 1 \qquad \dot{z} = 1$$





$$\dot{x} = \delta(z)(Sx - x) + \varepsilon f_\varepsilon(x, w, z)$$
$$\dot{w} = 1 + \varepsilon g_\varepsilon(x, w, z)$$
$$\dot{z} = 1$$

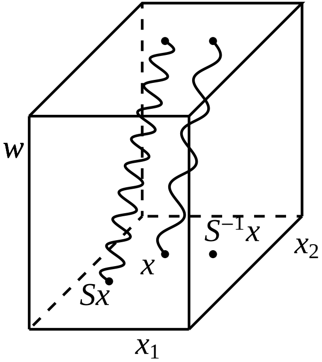


Look at Poincaré's map $\mathcal{S}$ at $t = 2\pi n$

$$x' = Sx + \varepsilon \bar{f}_\varepsilon(x, w)$$
$$w' = w + \varepsilon \bar{g}_\varepsilon(x, w)$$

a) volume preserving (if $\varepsilon = 0$ the $\mathcal{S}$ is not even ergodic & has one "central" Lyapunov exponent 0)

Th. *as close as wished (in C^2) to* $\bar{f} = \bar{g} = 0$ $\exists$ *open set of perturbations with* $\mathcal{S}_\varepsilon$



(1) Ergodic

(2) Central Lyap. exp. $\ell_\varepsilon > 0$

(3) $\exists$ $\mathcal{S}$*-invariant foliation* Λ *into* C^1*-smooth lines l*

(4) $\exists$*k and E of full vol. s.t.* $E \cap \ell$ *is exactly* $k < \infty$ *pts (!)*

[Conjectures: $k > 1$ & $k = 1$]

SW 1999, RW 2001

No synchronization: in (x, w, z) the planes $w = const$ are invariant under the P.-map but volatilize under perturbation

Is synch. possible in dissipation? which attractor structure?

1) Simplest possibility attractor="periodic orbit" on a orbit close to an unperturbed periodic one.

2) An attractor of "pathological nature" like the volume preserving cases but with Hausdorff dimension lower.

3) A periodic strange attractor: dissipation stabilizes a single one among the unperturbed invariant surfaces $w = const$

This can be easily tested in simulations: a variety of phenomena show up: consistent with 2) or 3). "Naivest" case

$$f = 0, \qquad g(x, w, z) = (\sin(z - w) + \sin(x_1 + z + w))$$

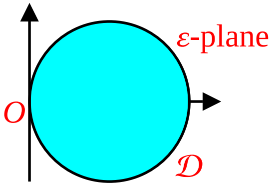
(first studied) immediately shows an instance of (3). Special:

$$g_0(x, w) \stackrel{def}{=} \int_0^{2\pi} g(x, w + t, t)dt \qquad = \sin w$$

$$g_1(x, w) \stackrel{def}{=} \int_0^{2\pi} \frac{\partial}{\partial w} g(x, w + t, t)dt = \cos w$$

Notice that in example $g_0(x, \pi) = 0,\ g_1(x, \pi) = \Gamma < 0,\ \forall x$.

Th. *If* $\exists\, w_0$ *such that* $g_0(x, w_0) = 0,\ g_1(x, w_0) = \Gamma < 0,\ \forall x$,
then $\exists \varepsilon_0$ for $0 < \varepsilon < \varepsilon_0$



(1) $\exists$ attractor $w(x) = w_0 + U_\varepsilon(x)$

(2) $U(x)$ is h-Hölder continuous

$$h \geq \frac{|\Gamma|}{\log \lambda_+}\varepsilon + O(\varepsilon^2)$$

(3) U_ε is analytic in ε in $\mathcal{D}$

Assumptions can be relaxed into

(a) $\int_0^{2\pi} \mathrm{d}t\, g(x, w_0 + t, t) = \varepsilon \overline{g}(x)$, for some $\overline{g}(x)$,

(b) $\int_0^{2\pi} \mathrm{d}t\, \partial_w g(x, w_0 + t, t) \leq \Gamma$, for $\Gamma < 0$,

Conjectures

(a’) $\int_0^{2\pi} \mathrm{d}t\, g(x, w_0 + t, t) = \widetilde{g}(x)$, with $\widetilde{g}(x)$ with 0 average,

(b’) $\int_0^{2\pi} \mathrm{d}t\, \partial_w g(x, w_0 + t, t) = \widetilde{g}_1(x)$, with $\widetilde{g}_1(x)$ with < 0 average.

Basics

$$(x(t), w(t), t) = (x, w + t + u(x, t), t), \qquad t \in (0, 2\pi]$$

$$\begin{aligned} \dot{x} &= \delta(z)(Sx - x) + \varepsilon f_\varepsilon(x, w, z) \\ \dot{w} &= 1 + \varepsilon g_\varepsilon(x, w, z) \\ \dot{z} &= 1 \end{aligned} \qquad \begin{aligned} u(x, 0) &= U(x) \\ u(x, 2\pi) &= U(Sx) \end{aligned}$$

Taylor expansion in u to second order

$$\begin{aligned} \dot{u}(x, t) &= \varepsilon g(x, w + t + u(x, t), t) \\ &\equiv \varepsilon g(x, w_0 + t, t) + \varepsilon \partial_w g(x, w_0 + t, t) u(x, t) + \varepsilon G(x, t, u(x, t)) \end{aligned}$$

Solved "as a linear equation" in terms of "Wronskian"

$$\Gamma(x, t, \tau) \overset{def}{=} \int_\tau^t \partial_w g(x, w + y, y) dy:$$

$$u(x, t) = e^{\varepsilon \Gamma(x,t,0)} u(x, 0) + \int_0^t e^{\varepsilon \Gamma(x,t,\tau)} \Big(\varepsilon g(x, w_0 + \tau, \tau) + \varepsilon G(x, \tau, u(x, \tau)) \Big) d\tau$$

Ideas: equations and invariance condition $(\Gamma(x, 2\pi, 0) \equiv \Gamma < 0)$

$$
\begin{aligned}
u(x,t) =& e^{\varepsilon\Gamma(x,t,0)}u(x,0) \\
&+ \int_0^t e^{\varepsilon\Gamma(x,t,\tau)}\Big(\varepsilon g(x, w_0+\tau,\tau) + \varepsilon G(x,\tau,u(x,\tau))\Big) \\
U(Sx) =& e^{\varepsilon\Gamma}U(x) \\
&+ \int_0^{2\pi} e^{\varepsilon\Gamma(x,2\pi,\tau)}\Big(\varepsilon g(x, w_0+\tau,\tau) + \varepsilon G(x,\tau,u(x,\tau))\Big)d\tau
\end{aligned}
$$

The assumptions $\Gamma < 0$ and $\int_0^{2\pi} g(x, w_0+t, t)dt = 0$ imply

$$
|u(\cdot, 2\pi)|_\infty < e^{\varepsilon\Gamma}|u(\cdot,0)|_\infty + O(\varepsilon^2) + O(\varepsilon|u|_\infty^2) \le e^{\frac{1}{2}\varepsilon\Gamma}|u|_\infty
$$

if $\frac{\delta}{2} < |u(x,0| < \delta$ with δ small and $0 \le \varepsilon \ll \delta$.

Hence there is an attractor in the slab $[w_0-\delta, w_0+\delta]$: but why is it a surface?

Next idea: replace some ε's with $\mu > 0$: let $U(x) = u(x,0)$

$$\begin{aligned} u(x,t) =& e^{\mu\Gamma(x,t,0)}u(x,0) + \\ & \int_0^t e^{\mu\Gamma(x,t,\tau)}\Big(\varepsilon g(x, w_0 + \tau, \tau) + \varepsilon G(x, \tau, u(x,\tau))\Big) \\ U(Sx) =& e^{\mu\Gamma}U(x) \\ & + \int_0^{2\pi} e^{\mu\Gamma(x,2\pi,\tau)}\Big(\varepsilon g(x, w_0 + \tau, \tau) + \varepsilon G(x, \tau, u(x,\tau))\Big) d\tau \end{aligned}$$

1) Fix μ small and prove existence of $U(x)$ analytic in $|\varepsilon| < C(\mu)$ complex.
2) Study $C(\mu)$ and show $C(\mu) > c\sqrt{\mu}$.
3) Conclude by $\varepsilon = \mu > 0$

Why Hölder continuity? convergence in ε reduces the question to first order. It is explicitly evaluated and shows the property of tiny Hölder continuity, $O(\varepsilon)$.

Define $\gamma_0(x,t) \stackrel{def}{=} g(x, w_0 + \tau, t)$

$$U(Sx) - \mathrm{e}^{\varepsilon\Gamma} U(x) = \varepsilon \int_0^{2\pi} \mathrm{d}\tau\, \mathrm{e}^{\varepsilon\Gamma(x,2\pi,\tau)} \Big(\gamma_0(x,\tau) + G(x,\tau,u(x,\tau))\Big),$$

and hence $U(x) = \mathrm{e}^{\varepsilon\Gamma} U(S^{-1}x) + F(S^{-1}x)$ and

$$F(x) = \varepsilon \int_0^{2\pi} \mathrm{d}\tau\, \mathrm{e}^{\varepsilon\Gamma(x,2\pi,\tau)} \Big(\gamma_0(x,\tau) + G(x,\tau,u(x,\tau))\Big)$$

$$U(x) = \sum_{k=1}^{\infty} \mathrm{e}^{\varepsilon\Gamma(k-1)} F(S^{-k}x),$$

The sum is $\sim \frac{1}{\varepsilon}$ but $F = O(\varepsilon^2)$ so $U \sim \varepsilon$.

Iterate to find higher order corrections.

$$u(x,t) = e^{\mu\Gamma(x,t,0)}U(x) + \xi(x,t)$$

$$\xi_1(x,t) = \int_0^t d\tau\, e^{\mu\Gamma(x,t,\tau)}\gamma_0(x,\tau), \qquad \xi_2(x,t) = 0\ ,$$

$$U_1(x) = \sum_{k=1}^{\infty} e^{\mu\Gamma(k-1)} \int_0^{2\pi} d\tau\, e^{\mu\Gamma(S^{-k}x,2\pi,\tau)}\gamma_0(S^{-k}x,\tau), \qquad U_2(x) = 0,$$

and, for $n \geq 3$,

$$\xi_n(x,t) = \int_0^t d\tau \sum_{p=2}^{\infty} \sum_{\substack{n_1,\dots,n_p \geq 1 \\ n_1+\dots+n_p=n-1}} e^{\mu\Gamma(x,t,\tau)} G_p(x,\tau) \prod_{i=1}^{p} \left(e^{\mu\Gamma(x,\tau,0)}U_{n_i}(x) + \xi_{n_i}(x,t)\right),$$

$$U_n(Sx) - e^{\mu\Gamma}U_n(x) = \int_0^{2\pi} d\tau \sum_{p=2}^{\infty} \sum_{\substack{n_1,\dots,n_p \geq 1 \\ n_1+\dots+n_p=n-1}} e^{\mu\Gamma(x,2\pi,\tau)} G_p(x,\tau) \cdot$$

$$\cdot \prod_{i=1}^{p} \left(e^{\mu\Gamma(x,\tau,0)}U_{n_i}(x) + \xi_{n_i}(x,t)\right),$$

Recursive definition of ξ_n, U_n.

$$U_n(x) = \sum_{k=1}^{\infty} e^{\mu\Gamma(k-1)} \int_0^{2\pi} \mathrm{d}\tau \sum_{p=2}^{\infty} \sum_{\substack{n_1,\dots,n_p \geq 1 \\ n_1+\dots+n_p=n-1}} \mathrm{e}^{\mu\Gamma(S^{-k}x, 2\pi, \tau)} G_p(S^{-k}x, \tau) \cdot$$

$$\cdot \prod_{i=1}^{p} \Big(\mathrm{e}^{\mu\Gamma(S^{-k}x, \tau, 0)} U_{n_i}(S^{-k}x, \tau) + \xi_{n_i}(S^{-k}x, \tau) \Big).$$