Ultraviolet and infrared properties of the sine-Gordon field

General problem: define, for $x \in R^2$ and $F = c\cos(\alpha\varphi_x)$

$$P(d\varphi) = const\,\exp[-\frac{1}{2}\int(\vec{\partial}_x\varphi_x)^2 + m^2\varphi_x^2 dx]$$
$$\cdot\exp[\lambda\int F(\varphi_x)dx]$$

Ultraviolet: $\int \to \int_\Lambda$, finite Λ, or $\Lambda = R^2, m > 0$

Infrared: $\int \to \int_{R^2}$ & $m = 0$

$\lambda = 0$: Gaussian field $\langle\varphi_x\varphi_y\rangle = \left(\frac{1}{-\Delta+m^2}\right)_{xy} =$

$$\begin{cases} \frac{1}{2\pi}\log\frac{1}{m|x-y|} & x-y \to 0, \quad e^{-m|x-y|} \quad x-y\to\infty \text{ UV} \\ const + \frac{1}{2\pi}\log\frac{1}{\mu|x-y|} & x-y\to\infty \quad \text{IR} \end{cases}$$

Regularization

$$C_{xy} \to C_{xy}^{[\leq N]} = \left(\frac{1}{-\Delta + 1} - \frac{1}{-\Delta + 2^{2N}} \right)_{xy}$$

$$= \begin{cases} \frac{1}{2\pi} \log \frac{1}{|x-y|} & |x-y| > 2^{-N} \\ \text{smooth} & |x-y| < 2^{-N} \end{cases} \qquad C_{00}^{[\leq N]} = \frac{N}{2\pi}$$

$$c_N = e^{\frac{\alpha^2}{4\pi} N} \Rightarrow$$

$$c_N \cos(\alpha\varphi) \equiv \sum_{k=0}^{\infty} \frac{(-\alpha^2)^k}{(2k)!} : \varphi^{2k} : \stackrel{def}{=} : \cos(\alpha\varphi) :$$

$$: \varphi^k : \stackrel{def}{=} \sqrt{\frac{C_{00}^{(\leq N)}}{2}}^{\,k} H_k\left(\frac{\varphi}{\sqrt{2C_{00}^{(\leq N)}}} \right)$$ “Wick’s polynomials”

$$P_{\alpha,\alpha,N}(d\varphi) = \frac{P_{0,N}(d\varphi)\exp[\lambda\int :\cos(\alpha\varphi_x^{(\leq N)}) : dx]}{Z(\alpha,\lambda,N)}$$

UV Theorem:

(1) $P_{\alpha,\lambda,N}(d\varphi) \xrightarrow[N\to\infty]{} P_{\alpha,\lambda}(d\varphi), \quad \alpha^2 < 8\pi$

(2) $Z(\alpha,\lambda,N) = Z_0(\alpha,\lambda,N)e^{\sum_{k=1}^{n} R_k(\alpha,N)\lambda^k},$

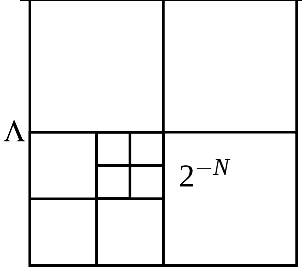
$$8\pi(1-\frac{1}{2n}) \leq \alpha^2 < 8\pi(1-\frac{1}{2(n+1)})$$

(3) $Z_0(\alpha,\lambda,N) \xrightarrow[N\to\infty]{} Z_0(\alpha,\lambda), \quad R_k(\alpha,N) \xrightarrow[N\to\infty]{} \infty$

(4) $: e^{\pm i\alpha\varphi_x} :, \qquad : e^{\varepsilon\varphi_x} :, \ |\varepsilon|^2 < 4\pi,$ good random v.

Comment: $: e_x^{\varepsilon\varphi^{(\leq n)}} := e^{-\frac{(\varepsilon^2 n)}{2\pi}} e^{\varepsilon\varphi_x^{(\leq n)}} \geq 0 \ \Rightarrow \ \forall n \leq \infty$

$\mu(dx) = \frac{:e^{\varepsilon\varphi_x^{(\leq n)}}:}{\int_\Lambda :e^{\varepsilon\varphi_v^{(\leq n)}}: dv} dx$ **is a random measure on** Λ



Idea: *Hierarchical model (Dyson-Wilson)*

Pavements $Q_k \subset \Lambda$ side, Δ's size 2^{-k}

$z_\Delta \to \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_\Delta^2} dz_\Delta$, iid

$\varphi_x = \sum_{\Delta \ni x} z_\Delta$

$\varphi_x^{(\leq N)} = \sum_{k=1}^N \sum_{Q_k \ni \Delta \ni x} z_\Delta$

$$\int P^{(\leq N)}(d\varphi^{\leq N}) \cdot = \int P^{(<N)}(d\varphi^{<N}) \int P_N(dz) \cdot$$

$$\Delta \in Q_N, \Delta' \supset \Delta \;\Rightarrow\; \left(\int g(dz) e^{\lambda |\Delta| :\cos(\alpha(\varphi^{(<N)}+z)):} \right)^4$$

$\text{size} = \lambda\, 2^{-2N} 2^{\frac{\alpha^2}{4\pi}N}$, $\qquad (size)^2 = \lambda^2 (2^{-2N} 2^{\frac{\alpha^2}{4\pi}N})^2$

$$\int g(dz) e^F = e^{4\int g(dz)F + error(size)^2}$$

number errors= $|\Lambda| 2^{2N} \Rightarrow$ total $(2^{-2N} 2^{\frac{\alpha^2}{4\pi}N})^2 2^{2N} |\Lambda|$ summable!

$$\int P^{(N)}(d\varphi^{(N)})\, e^{-\lambda \int :\cos(\alpha\varphi_x^{(N)}):\, dx}$$

$$= e^{\int P^{(N)}(d\varphi^{(N)}) :\cos(\alpha\varphi_x^{(N)}):\, dx} e^{\pm\lambda^2 (2^{-2N} 2^{\frac{\alpha^2}{4\pi}N})^2 2^{-2N} |\Lambda|}$$

Key property of Wick's polynomials $\int P(d\zeta) : (\varphi+\zeta)^k :=: \varphi^k :$

$$\Rightarrow \int P^{(N)}(d\varphi^{(N)}) : \cos(\alpha(\varphi_x^{(<N)} + \varphi_x^{(N)})) : dx =: \cos(\alpha\varphi_x^{(<N)}) :$$

Conclusion for $\alpha^2 < 4\pi$

$$\int P^{(N)}(d\varphi^{(N)}) e^{-\lambda \int :\cos(\alpha\varphi_x^{(N)}):\, dx} = e^{\pm \sum_{k=0}^{\infty} \lambda^2 (2^{-2k} 2^{\frac{\alpha^2}{4\pi}k})^2 2^{-2k} |\Lambda|}$$

Beyond 4π? expansion to order n: error becomes summable when $(2^{-2N} 2^{\frac{\alpha^2}{4\pi}N})^n 2^{-2N} < 1$ i.e. if $\alpha^2 < 8\pi(1-\frac{1}{2n})$.

However calculations to order n are harder (yet possible)

Important that the free energy defintion only requires constant counterms (the $R_k(\alpha,N)$): physical intepretation possible (Yukawa-gas)

Infrared: Coulomb and dipoled gases

$$C_{xy}^{(-R)} = \frac{1}{(2\pi)^2}\int d^2k\,(\frac{1}{k^2+2^{-2R}} - \frac{1}{k^2+1})e^{ik(x-y)}$$

$$\lim_{R\to\infty}(C_{xy}^{(-R)} - C_{00}^{(-R)}) = W(x-y) \simeq \frac{1}{2\pi}\log|x-y|^{-1}$$

$$C_{00}^{(-R)} = \frac{R}{2\pi}\log 2,$$

$$\lim_{R\to\infty} Z^{(-R)}(I,\beta,\lambda) = \lim_{R\to\infty}\int P(d\psi^{(\geq -R)})e^{\lambda\int_I \cos\alpha\psi_x^{(-R,0)}\,dx}$$

is the partition function of the neutral Coulomb gas ($\alpha^2 = \beta e^2$)

The change of variables $\varphi_x^{(<R)} = \psi_{2^R x}^{(\geq -R)}$ transforms the problem into an UV problem in a box $I2^{-R}$

$$Z^{(-R)}(I,\beta,\lambda) = \int P(d\varphi^{(\leq R)} e^{\frac{\lambda}{2} 2^{(2-\frac{\alpha^2}{4\pi})R} \sum_{\sigma=\pm} \int_{I2-R} :e^{i\sigma\alpha\varphi_x^{(\leq -R)}}: dx}$$

Theorem For $\alpha_n^2 \stackrel{def}{=} 8\pi(1-\frac{1}{2n}) \leq \alpha^2 < 8\pi(1-\frac{1}{2(n+1)})$ the first $2n$ coefficients of the Mayer expansion are finite. Or

(1) as the temperature decreases for $+\infty$ (i.e. $\alpha^2 = 0$) to $T_c = \frac{1}{\sqrt{8\pi}}$ the Mayer series coefficients become defined and

(2) for $T < T_c$ all are defined (BGM).

(3) For α^2 large enough $\alpha^2 > 24\pi$ the Mayer series converges (at small λ) (GawKup).

(4) At high T: "plasma phase" (Debye screening, (Bry))

(5) At low temperature "multipole phase" or "Kosterlitz-Thouless" phase

Hierarchical model

In this case it can be shown

(1) the Mayer series converges for $\alpha^2 > 8\pi$, i.e. in the entire multipole phase, (BGN)

(2) correlations have power law decay exponent $\frac{\alpha^2}{2\pi}$ (BGN)

Furthermore the “dipole gas” can be represented via the integral

$$\int P(d\psi^{(-R)}) = const e^{\lambda \int_I dx d\theta \cos(\alpha \partial \varphi_x \cdot \vec{n}(\theta))}$$

and in this case it has been proved convergence of the Mayer expansion (GawKup) at all α^2 with power law decay of the corelations. With our method we can only prove finiteness of all Mayer coefficients (BGN).

The hierarchical version is obtained by replacing z_Δ with $\pm z_\Delta$ depending on the position of the point $x \in \Delta$ in the left or right half. For this model: convergence of the Mayer expansion

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