Formal perturbation analysis of a non equilibrium stationary state

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Non-equilibrium: statistics is often shown to exist.

(By) “compactness methods”: very unsatisfactory.

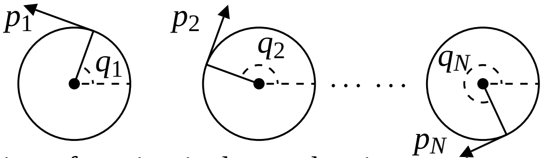
Dissatisfaction: when physical quantities are needed “no answers”.

E.g. for hard spheres systems continuity of correlation of a stationary nonequilibrium distibution is not known $\Rightarrow$ no answer to simple questions like $\langle \partial_q \rho(q-q') \rangle$.

Particularly valuable are therefore exactly soluble models.

Unfortunately they are very few , most involve stochastic forces.

Even so the study is very difficult. Here the question of "computing" correlations for a "trivial" system. The $N = 1$ **case**



The equation of motion is the stochastic equation

$$\dot{q} = \frac{p}{J} \qquad \dot{p} = -\partial U - \tau - \frac{\xi}{J} p + \sqrt{\frac{2\xi}{J}}\,\dot{\mathbf{w}}$$

$\dot{\mathbf{w}}$ w.n. of width $(\frac{J}{\beta_0\, dt})^{\frac{1}{2}}$,
$U = -gV\cos(q)$ conservative force, τ torque, ξ friction, J inertia

β_0 inverse temperature

Problem: find the stationary state distribution

It is hard to believe that this in not known **???**.

$F(p,q)dpdq$: the stochastic equation yields the PDE $\mathscr{L}^*F = 0$

$$\begin{aligned}\mathscr{L}^*F = & -\left\{\left(\frac{p}{J}\partial_q F(q,p) - (\partial_q U(q) + \tau)\partial_p F(q,p)\right)\right. \\ & \left. -\xi\left(\beta_0^{-1}\partial_p^2 F(q,p) + \frac{1}{J}\partial_p(pF(q,p))\right)\right\}\end{aligned}$$

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$$\mathscr{L}^*F = -\Big\{ \Big(\frac{p}{J}\partial_q F(q,p) - (\partial_q U(q) + \tau)\partial_p F(q,p) \Big) - \xi \Big(\beta_0^{-1}\partial_p^2 F(q,p) + \frac{1}{J}\partial_p (pF(q,p)) \Big) \Big\}$$

General results: **most interesting: simulations, ILOS1: [1]**

(1) There exists a smooth solution (Hormander)

(2) is exponentially approached by initial δ (Mattingly-Stuart)

(3) it is positive $F(q,p) = \frac{e^{-\frac{\beta}{2J}p^2}\rho(p,q)}{\sqrt{2J\beta^{-1}}} = G(p)\rho(p,q)$ (MS)

(4) $\int G(p)\rho(p,q)^2 < \infty$, (???)

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(5) $\Rightarrow$ if $: p^n := \left(\frac{J\beta^{-1}}{2}\right)^{\frac{n}{2}} H_n(\frac{p}{\sqrt{2J\beta^{-1}}})$ (Wick,Hermite poly.)

$\rho(p,q) = G(p)(\rho_0(q) + \rho_1(q)p + \rho_2(q) :p^2: + \rho_3(q) :p^3: + \dots)$

Problem: Find the expansion in g of $\rho_n(q)$ (why care?)

By algebra $\mathscr{L}^* F = 0$ becomes: $n \geq 0$ $(\rho_{<0} \equiv 0)$

$$n\beta^{-1}\partial\rho_n(q) + \Big[\frac{1}{J}\partial\rho_{n-2}(q) + \frac{\beta}{J}(\partial U(q) + \tau)\rho_{n-2}(q)$$
$$+ (n-1)\frac{\xi}{J}\rho_{n-1}(q)\Big] = 0$$

Messy! compute the $\rho_n^{[1]}(q)$ and go dimensionless with

$$\sigma_n(q) \stackrel{def}{=} \rho_n(q)\xi^n n!, \quad \eta \stackrel{def}{=} \beta\xi^2/J, \beta\tau, \quad \beta V$$

(1) Recursion is linear: take the F.T. $\sigma_{n,k}$

(2) Recursion is second degree: take $\mathbf{Z}_{n,k} \stackrel{def}{=} \binom{\sigma_{n,k}}{\sigma_{n-1,k}(q)}$

for $k \neq 0$ Fourier modes of the $\sigma's$

(3) u_n = average of σ_n; then $(r = 1)$

$$\mathbf{Z}^{[r]}_{n,k} = M_{n,k}\mathbf{Z}^{[r]}_{n-1,k} + \mathbf{X}^{[r]}_{n,k}, \qquad \mathbf{Z}^{[r]}_{2,k} = \mathbf{Y}^{[r]}_k$$
$$u^{[r]}_n = -\beta\tau u^{[r]}_{n-1} + v^{[r]}_n, \qquad u^{[r]}_2 = u^{[r]}$$

This can be written explicitly: the first order for $n > 2$

$$\mathbf{Z}_{n,k} = M_{n,k}\mathbf{Z}_{n-1,k} + \mathbf{X}_{n,k}, \qquad \mathbf{Z}^{[r]}_{2,k} = \mathbf{Y}^{[r]}_{k}, \qquad \mathbf{X}^{[1]}_{n,k} = u_n|2\rangle$$

$$u_n = -\beta\tau u_{n-1}, \qquad u_2 = (-\beta\tau)^2$$

so $u_n = (-\beta\tau)^n$, and with initial data at $n = 2$. Set $m \equiv n-1$

$$a_k \stackrel{def}{=} (1 - i\frac{\beta\tau}{k})i, \quad |1\rangle \stackrel{def}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle \stackrel{def}{=} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$M_{n,k} \stackrel{def}{=} \begin{pmatrix} \frac{m}{k}i\eta & ima_k\eta \\ 1 & 0 \end{pmatrix}, \; M^{-1}_{n,k} \stackrel{def}{=} \begin{pmatrix} 0 & 1 \\ \frac{1}{ima_k\eta} & -\frac{1}{ka_k} \end{pmatrix},$$

$$\mathbf{Y}_1 \stackrel{def}{=} \left(-\eta a_k\sigma_{0,1} - \eta\beta V\right)|1\rangle \stackrel{def}{=} \sigma_{0,1}\begin{pmatrix} \overline{y}^{[1]}_k \\ 0 \end{pmatrix} - \eta\beta V|1\rangle$$

Conclusion: $\sigma_{0,k}$ determines everything at given order: keeping in mind that at order r it must be $|k| \le r$.

So infinitely many solutions! however $\rho(q,p)$ must be L_2:

Required: $|\sigma_n(q)| \le \frac{\varepsilon^n}{\sqrt{n!}}$. **Formally**

$$\mathbf{Z}_n = \lambda M_{n+1}^{-1} \dots M_N^{-1} \dots . + \sum_{h=n+1}^{\infty} M_{n+1}^{-1} \dots M_h^{-1} \mathbf{X}_h$$

A cut-off version is

$$\mathbf{Z}_n(N) = \lambda \lambda_N M_{n+1}^{-1} \dots M_N^{-1} |2\rangle + \sum_{h=n+1}^{N-n} M_{n+1}^{-1} \dots M_h^{-1} \mathbf{X}_h$$

Need a theorem: for $|\beta\tau|, \eta^{-1}$ small enough

Theorem 1: *Given k, if η is large enough there is a sequence $\Lambda_k(n_1, N)$ such that for all $n_1 \ge 2$ and all $k \ne 0$ the*

$$\zeta_k(n_1, N) \stackrel{def}{=} \begin{pmatrix} \zeta_k(n_1, N)_1 \\ 1 \end{pmatrix} = \Lambda_k(n_1, N) M_{n_1+1}^{-1} \dots M_N^{-1} |2\rangle \quad (0.1)$$

is s.t. $|\zeta_k(n_1, N)_1| \le 1$ *and* $\zeta_k(n_1) \stackrel{def}{=} \lim_{N\to\infty} \zeta_k(n_1, N)$ *exists.*

Corollary: *A solution, unique exponentially, is*

$$\mathbf{Z}_n = \lambda\,\zeta_n + \xi_n, \qquad n \geq 2, \qquad \text{where}$$

$$\zeta_n = M_n \cdots M_3\,\zeta(2), \qquad \xi_n = \sum_{h=n+1}^{\infty} -\beta V \eta M_{n+1}^{-1} \ldots M_h^{-1} \mathbf{X}_k^{[1]}$$

$$|\zeta_n| \leq B|1 - i\beta\tau|^n, \qquad |\mathbf{Z}_n| \leq B\beta V|1 - i\beta\tau|^n$$

Once $\zeta(2) = \lim_{N\to\infty} M_3^{-1} \cdots M_N^{-1} |2\rangle$ known impose initial data

$$\mathbf{Z}_2 \equiv (\sigma_{0,1}^{[1]} \overline{y}_1^{[1]} - \eta\beta V)\,|1\rangle = \lambda\,\zeta_2 + \xi_2$$

(recall $\overline{y}^{[1]} = -\eta(1 + \frac{\beta\tau}{i})$) or

$$\sigma_{0,1}^{[1]} \overline{y}_1^{[1]} - \lambda\,\zeta(2)_1 = (\xi(2)_1 + \eta\beta V)$$
$$-\lambda = \xi(2)_2$$

determines the only unknown $\sigma_{0,1}^{[1]}$.

Hence the key is the theorem

Once more this is a problem on a Ising system controlled by γ

$$\gamma = -\frac{4ia_k k^2}{m\eta}, \qquad \lambda_{m,k,\pm} \stackrel{def}{=} -\frac{1 \pm \sqrt{1+\gamma}}{2\,k\,a_k}$$

and spectral decomposition of M_n^{-1}:

$$M_{n,k}^{-1} = \sum_{\sigma=\pm1} \lambda_{n,k,\sigma} \frac{|v_{n,\sigma}\rangle\langle w_{n,\sigma}|}{\langle w_{n,\sigma}|v_{n,\sigma}\rangle},$$

$$|v_{n,\sigma}\rangle = \lambda_{n,\sigma}^{-1} \begin{pmatrix} 1 \\ \lambda_{n,\sigma} \end{pmatrix},$$

$$\langle w_{n,\sigma}| = \left(\varepsilon_n, \lambda_{n,\sigma}\right), \qquad \varepsilon_n = \frac{\gamma}{4k^2 a_k^2}$$

The Ising model arises because $M_n^{-1} \cdots M_N^{-1} |2\rangle$ is writing explicitly the matrix product $M_n^{-1} \cdots M_N^{-1} |2\rangle$ as

$$\sum_{\sigma_3, \ldots, \sigma_N} |v_{3,\sigma_3}\rangle \cdot \Big(\prod_{j=3}^{N} \lambda_{j,\sigma_j} \Big) \prod_{j=3}^{N} \frac{\langle w_{j,\sigma_j} | v_{j+1,\sigma_{j+1}} \rangle}{\langle w_{j,\sigma_j} | v_{j,\sigma_j} \rangle}$$

"expectation" of $|v_{3,\sigma_3}\rangle$ in a Gibbs state. Gibbs factor?

Spin configuration = intervals of $-$ separated $+$, then

$$\overline{\Lambda}_N \prod_{i=1}^{p} I_J \qquad \text{is the Gibbs weight}$$

$$\rho(J) \stackrel{def}{=} \frac{\sum_{p \geq 1} \sum^{*}_{J_1 < \ldots < J_p} \Big(\prod_{s=1}^{p} I_{J_s} \Big)}{\Omega_2(N)}$$

$$P_- \stackrel{def}{=} \sum_{\{3\} \in J} \rho(J), \ \zeta(2) = \lambda_{3,-}^{-1} P_- + \lambda_{3,+}^{-1} P_+$$

$$\overline{\Lambda}_N^{[2]} \stackrel{def}{=} \Big(\prod_{j=3}^{N} \lambda_{j,+}\Big) \Big(\prod_{j=3}^{N} \frac{\langle w_{j,+} | v_{j+1,+}\rangle}{\langle w_{j,+} | v_{j,+}\rangle}\Big)$$

$$I_J \stackrel{def}{=} \Big(\prod_{j=k}^{k'} \frac{\lambda_{j,-}}{\lambda_{j,+}}\Big) \Big(\prod_{j=k}^{k'} \frac{\langle w_{j,-} | v_{j+1,-}\rangle}{\langle w_{j,-} | v_{j,-}\rangle} \frac{\langle w_{j,+} | v_{j,+}\rangle}{\langle w_{j,+} | v_{j+1,+}\rangle}\Big)$$

$$\cdot \frac{\langle w_{k-1,+} | v_{k,-}\rangle}{\langle w_{k-1,+} | v_{k,+}\rangle} \frac{\langle w_{k',-} | v_{k'+1,+}\rangle}{\langle w_{k',+} | v_{k'+1,+}\rangle} \frac{\langle w_{k',+} | v_{k',+}\rangle}{\langle w_{k',-} | v_{k',-}\rangle}$$

$$|I_J| \le W_J \stackrel{def}{=} \Big(\frac{\sqrt{2}}{\eta}\Big)^{|J|} \Big(\prod_{j\in J} \frac{1}{j-1}\Big) \frac{2k^4}{\eta^2 (k'-1)^3}$$

Non transl. inv. but very small already for J close to the origin.

Hence the $P_\pm$ can be computed as convergent series in I_J's as well as the partition function $exp(\Omega_N)$ which amits a limit as $N \to \infty$ (no logarithm necessary).

Theorem: **Fixed $r > 0$ the equation $\mathscr{L}^* F(p,q) = 0$**

(1) admits (at most) a 1-que C^r (in g,p,q) solution nonnegative and in $L_1(dpdq)$ for τ, g, ξ^{-1} small.

(2) The coefficients $F^{[r]}(p,q)$ of its Taylor expansion at $g = 0$ are uniquely determined, analytic in q,p,

(3) are explicitly computable.

Conjecture: $r = \infty$ and F analytic in g.

The idea should be applicable to the evaluation of the Lyapunov exponents of infinite products of matrices close to a hyperbolic matrix.

In particular could yield analyticity of the leading exponent in terms of parameters defining the matrices (a very special case of a result by Ruelle).

A. Iacobucci, F. Legoll, S. Olla, and G. Stoltz.
Negative thermal conductivity of chains of rotors with mechanical forcing.
Physical Review E, 84:061108 +6, 2011.