Formal perturbation analysis of a non equilibrium stationary state

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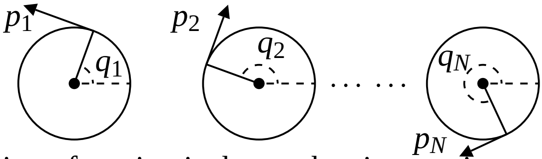
Non-equilibrium: statistics is often shown to exist: "compactness methods": very unsatisfactory.

Dissatisfaction: when needed: "no answers".

E.g. for hard spheres systems regularity of correlation of a stationary nonequilibrium distibution is not known $\Rightarrow$ no answer to simple questions like $\langle \partial_q \rho(q-q') \rangle$.

Unfortunately very few exact, most involve stochastic forces.

Even so the study is very difficult. Here the question of "computing" correlations for a "trivial" system. The $N = 1$ **case**



The equation of motion is the stochastic equation

$$\dot{q} = \frac{p}{J} \qquad \dot{p} = -\partial U - \tau - \frac{\xi}{J} p + \sqrt{\frac{2\xi}{J}}\, \dot{\mathbf{w}}$$

$\dot{\mathbf{w}}$ w.n. of width $\left(\frac{J}{\beta_0\, dt}\right)^{\frac{1}{2}}$,
$U = -gV\cos(q)$ conserv. force, τ torque, ξ friction, J inertia

β_0 noise inverse temperature

Problem: find the stationary state distribution for $\mathbf{N = 1}$

It is hard to believe that this in not known **???**.

$F(p,q)dpdq$: the stochastic equation yields the PDE $\mathscr{L}^*F = 0$

$$\begin{aligned}\mathscr{L}^*F = & -\Big\{\Big(\frac{p}{J}\partial_q F(q,p) - (\partial_q U(q)+\tau)\partial_p F(q,p)\Big) \\ & -\xi\Big(\beta_0^{-1}\partial_p^2 F(q,p) + \frac{1}{J}\partial_p(pF(q,p))\Big)\Big\}\end{aligned}$$

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$$\mathscr{L}^* F = -\Big\{ \Big(\frac{p}{J} \partial_q F(q,p) - (\partial_q U(q) + \tau) \partial_p F(q,p) \Big) - \xi \Big(\beta_0^{-1} \partial_p^2 F(q,p) + \frac{1}{J} \partial_p (pF(q,p)) \Big) \Big\}$$

Gen. results: **most interesting are simulations, ILOS1: [1]**
but "no" theory but

(1) There exists a smooth solution   (Hormander)

(2) it is positive $F(q,p) = \frac{e^{-\frac{\beta}{2J}p^2} \rho(p,q)}{\sqrt{2J\beta^{-1}}} = G(p)\rho(p,q)$ (MS)

(3) is exponentially approached by initial δ (Mattingly-Stuart)

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$$\mathscr{L}^* F = -\Big\{\Big(\frac{p}{J}\partial_q F(q,p) - (\partial_q U(q)+\tau)\partial_p F(q,p)\Big) - \xi\,\Big(\beta_0^{-1}\partial_p^2 F(q,p) + \frac{1}{J}\partial_p(pF(q,p))\Big)\Big\}$$

General results: **most interesting: simulations, ILOS1: [1]**

(1) There exists a smooth solution (Hormander)

(2) is exponentially approached by initial δ (Mattingly-Stuart)

(3) it is positive: $\frac{e^{-\frac{\beta}{2J}p^2}\rho(p,q)}{\sqrt{2J\beta^{-1}p^2}} = G(p)\rho(p,q)$

Guess if $: p^n := \left(\frac{J\beta^{-1}}{2}\right)^{\frac{n}{2}} H_n(\frac{p}{\sqrt{2J\beta^{-1}}})$ (Wick,Hermite poly.)

$\rho(p,q) = G(p)(\sum_{n=0}^{\infty}\rho_n(q)\,:p^n:)$, **Problem:** Find it

By algebra $\mathscr{L}^* F = 0$ becomes: $n \geq 0$ $(\rho_{<0} \equiv 0)$

$$n\beta^{-1}\partial\rho_n(q) + \Big[\frac{1}{J}\partial\rho_{n-2}(q) + \frac{\beta}{J}(\partial U(q) + \tau)\rho_{n-2}(q)$$
$$+ (n-1)\frac{\xi}{J}\rho_{n-1}(q)\Big] = 0$$

Go dimensionless: $\sigma_n(q) \stackrel{def}{=} \rho_n(q)\xi^n n!, \eta \stackrel{def}{=} \beta\xi^2/J, \beta\tau, \quad \beta V$

(1) Recursion is linear: take the F.T. $\sigma_{n,k}$

(2) Recursion is second degree: take $\mathbf{Z}_{n,k} \stackrel{def}{=} \binom{\sigma_{n,k}}{\sigma_{n-1,k}(q)}$

$$\mathbf{Z}^{[r]}_{n,k} = M_{n,k}\mathbf{Z}^{[r]}_{n-1,k} + \mathbf{X}^{[r]}_{n,k}, \qquad \mathbf{Z}^{[r]}_{2,k} = \mathbf{Y}^{[r]}_{k}$$
$$u^{[r]}_{n} = -\beta\tau u^{[r]}_{n-1} + v^{[r]}_{n}, \qquad u^{[r]}_{2} = u^{[r]}$$

Conclusion: $\sigma^{[r']}_{0,k}, r' < r$ and $\sigma^{[r]}_{0,k}$ determines everything at given order: (at order r it must be $|k| \leq r$.)

Conclusion: So infinitely many solutions!

However $\rho(q,p)$ must be $L_2(G)$ (Olla)

Requirement (at least): $|\sigma_n(q)| \le \frac{c^n}{\sqrt{n!}}$. **Formal solution**

$$\mathbf{Z}_n = \lambda M_{n+1}^{-1} \ldots M_N^{-1} \ldots . + \sum_{h=n+1}^{\infty} M_{n+1}^{-1} \ldots M_h^{-1} \mathbf{X}_h$$

Infinite matrix product: for η^{-1} small enough, let $|2\rangle \stackrel{def}{=} \binom{0}{1}$

Key: *Given k, if η is large enough there is a sequence $\Lambda_k(n_1,N)$ such that for all $n_1 \ge 2$ and all $k \ne 0$ the*

$$\zeta_k(n_1,N) \stackrel{def}{=} \begin{pmatrix} \zeta_k(n_1,N)_1 \\ 1 \end{pmatrix} = \Lambda_k(n_1,N) M_{n_1+1}^{-1} \ldots M_N^{-1} |2\rangle$$

is s.t. $|\zeta_k(n_1,N)_1| \le |k|$ *and* $\zeta_k(n_1) \stackrel{def}{=} \lim_{N\to\infty} \zeta_k(n_1,N)$ *exists.*

Once more this is a problem on a Ising system controlled by γ

$$\gamma = -\frac{4ia_k k^2}{m\eta}, \qquad \lambda_{m,k,\sigma} \stackrel{def}{=} -\frac{1+\sigma\sqrt{1+\gamma}}{2\,k\,a_k},\ \sigma = \pm 1$$

$a_k \stackrel{def}{=} i(1-\frac{i\beta\tau}{k})$ and spectral decomposition of M_n^{-1}:

$$M^{-1}_{n,k} = \sum_{\sigma=\pm 1} \lambda_{n,k,\sigma} \frac{|v_{n,\sigma}\rangle\langle w_{n,\sigma}|}{\langle w_{n,\sigma}|v_{n,\sigma}\rangle},$$

$$|v_{n,\sigma}\rangle = \begin{pmatrix}\lambda^{-1}_{n,\sigma}\\ 1\end{pmatrix},$$

$$\langle w_{n,\sigma}| = \Big(\varepsilon_n, \lambda_{n,\sigma}\Big), \qquad \varepsilon_n = \frac{\gamma}{4k^2 a_k^2}$$

The Ising model arises because $M_n^{-1}\cdots M_N^{-1}|2\rangle$ is writing explicitly the matrix product $M_n^{-1}\cdots M_N^{-1}|2\rangle$ as

$$\sum_{\sigma_3,\ldots,\sigma_N} |v_{3,\sigma_3}\rangle \cdot \Big(\prod_{j=3}^{N}\lambda_{j,\sigma_j}\Big)\prod_{j=3}^{N}\frac{\langle w_{j,\sigma_j}|v_{j+1,\sigma_{j+1}}\rangle}{\langle w_{j,\sigma_j}|v_{j,\sigma_j}\rangle}$$

"expectation" of $|v_{3,\sigma_3}\rangle$ in a Gibbs state. Gibbs factor?

Spin configuration = intervals of $-$ separated $+$, then

$$\overline{\Lambda}_N \prod_{i=1}^{p} I_J \qquad \text{is the Gibbs weight}$$

$$\rho(J) \stackrel{def}{=} \frac{\sum_{p\geq 1}\sum^{*}_{J_1<\ldots<J_p}\Big(\prod_{s=1}^{p} I_{J_s}\Big)}{\Omega_2(N)}$$

$$P_- \stackrel{def}{=} \sum_{\{3\}\in J}\rho(J),\ \zeta(2) = \lambda_{3,-}^{-1}P_- + \lambda_{3,+}^{-1}P_+$$

$$\overline{\Lambda}_N^{[2]} \stackrel{def}{=} \Big(\prod_{j=3}^{N} \lambda_{j,+}\Big)\Big(\prod_{j=3}^{N} \frac{\langle w_{j,+}|v_{j+1,+}\rangle}{\langle w_{j,+}|v_{j,+}\rangle}\Big)$$

$$I_J \stackrel{def}{=} \Big(\prod_{j=k}^{k'} \frac{\lambda_{j,-}}{\lambda_{j,+}}\Big)\Big(\prod_{j=k}^{k'} \frac{\langle w_{j,-}|v_{j+1,-}\rangle}{\langle w_{j,-}|v_{j,-}\rangle}\,\frac{\langle w_{j,+}|v_{j,+}\rangle}{\langle w_{j,+}|v_{j+1,+}\rangle}\Big)$$

$$\cdot \frac{\langle w_{k-1,+}|v_{k,-}\rangle}{\langle w_{k-1,+}|v_{k,+}\rangle}\,\frac{\langle w_{k',-}|v_{k'+1,+}\rangle}{\langle w_{k',+}|v_{k'+1,+}\rangle}\,\frac{\langle w_{k',+}|v_{k',+}\rangle}{\langle w_{k',-}|v_{k',-}\rangle}$$

$$|I_J| \le W_J \stackrel{def}{=} \Big(\frac{\sqrt{2}}{\eta}\Big)^{|J|}\Big(\prod_{j\in J}\frac{1}{j-1}\Big)\frac{2k^4}{\eta^2(k'-1)^3}$$

Non transl. inv. but very small already for J close to the origin.

Hence the $P_\pm$ can be computed as convergent series in I_J's as well as the partition function $exp(\Omega_N)$ which amits a limit as $N \to \infty$ (no logarithm necessary).

Fixed R then if viscosity ξ is large enough and $0 \le r \le R$

Theorem: The equation $\mathscr{L}^* F(p,q) = 0$

(1) admits (at most) a 1-que C^r (in g) solution nonnegative and in $L_1(dpdq)$ for g small.

(2) the coefficients $F^{[r]}(p,q)$ of its Taylor expansion at $g = 0$ are uniquely determined, analytic in q, p,

(3) are explicitly computable.

Conjecture: The coefficients exist for all r

Question: is F analytic in g because $\exists A, C, c$ s.t.

$$|\rho_{n,k}^{[r]}| \le A^r C^n e^{-c|k|} \frac{1}{n!} \qquad ? \qquad C = \overline{C}|k|?$$

The idea should be applicable to the evaluation of the Lyapunov exponents of infinite products of matrices close to a hyperbolic matrix.

In particular could yield analyticity of the leading exponent in terms of parameters defining the matrices (a very special case of a result by Ruelle).

A. Iacobucci, F. Legoll, S. Olla, and G. Stoltz.
Negative thermal conductivity of chains of rotors with mechanical forcing.
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