Vibrating string

$$\begin{aligned} \partial_t^2 u =& c^2 \partial_x^2 u, \qquad x \in [0,a],\ u(a) = u(b) = 0 \\ u(x,t) =& \varphi(x-ct) + \varphi(x+ct) \end{aligned}$$

D'Alembert: $\varphi \in \mathscr{C}_2(R)$, $2a$-periodic, odd around 0 and a
Euler: $\varphi \in \mathscr{C}_1(R)$, $2a$-periodic, odd around 0 and a
Other (Taylor, D.Bernoulli)

$$u(x) = \sum_n \alpha_n \sin(\frac{2\pi}{2a} nx) \cos(\frac{2\pi}{2a} nct)$$

according to Euler this is the same as D'Alembert.
Problem: each derived in its own way, eg D. Bernoulli"
L'Auteur déduit cette ingénieuse théorie par une espèce d'induction qu'il tire de la considération des mouvements d'un nombre de corps qui sont supposés former des vibrations régulières et isochrones ...

Lagrange: back to first principles.
New approach: think the string as collection of particles

Il resulte de tout cet exposé que l'Analyse que nous avons proposée dans le Chapitre précédent est peut-être, la seule qui puisse jeter sur ces matières obscures une lumière suffisante à éclaircir les doutes qu'on forme de part et d'autre.

notices that the problem is diagonalization of a $(m-1)\times(m-1)$ tridiagonal symmetric matrix.

.. je ne crois pas qu'on ait jamais donné pour cela une formule générale, telle que nous venons de la trouver.

This corresponds to the Lagrangian

$$\mathscr{L} = \frac{\mu}{2}\delta \sum_{n\delta\in\Lambda_0}\left(\dot{\varphi}_{n\delta}^2 - c^2\sum_{j=1}^{D}(\varphi_{n\delta+e_j\delta}-\varphi_{n\varepsilon})^2\right)$$

$$\varphi^{(\delta)}(\xi,t)=\sum_{h=1}^{m-1}\left\{\widetilde{A}_h\sqrt{\frac{2}{m}}\sin\frac{\pi h}{a}\xi\cdot\cos\omega_h t+\widetilde{B}_h\sqrt{\frac{2}{m}}\sin\frac{\pi h}{a}\xi\cdot\sin\omega_h t\right\}$$

is the solution

$$\sqrt{\frac{2}{m}}\widetilde{A}_h=\frac{2}{m}\sum_{i=1}^{m-1}\left(\sin\frac{\pi h}{a}\xi\right)Z(\xi)\xrightarrow[\delta\to 0]{}\frac{2}{a}\int_0^a Z(x)\left(\sin\frac{\pi h}{a}x\right)dx$$

$$\sqrt{\frac{2}{m}}\widetilde{B}_h=\text{similar},\qquad \omega_h=c\sqrt{2\frac{1-\cos(\frac{\pi h\delta}{a})}{\delta^2}}\xrightarrow[\delta\to 0]{}\overline{\omega}_h=c\frac{\pi h}{a}$$

formally converging to

$$u(x,t)=\sum_{h=0}^{\infty}\sin\frac{\pi h}{a}x\Big\{\Big(\frac{2}{a}\int_0^a Z(x')\sin\frac{\pi h}{a}x'\,dx'\Big)\cos\overline{\omega}(h)t$$
$$+\Big(\frac{2}{a}\int_0^a U(x')\sin\frac{\pi h}{a}x'\,dx'\Big)\frac{\sin\overline{\omega}(h)t}{\overline{\omega}(h)}\Big\}$$

A few critiques induced L. to write responses

$$\cos x + \cos 2x + \cos 3x + \ldots = -\frac{1}{2} \qquad ???$$

il ne sera pas hors de propos de démontrer encore la même proposition d'une autre manière. Or

$$\sin \pi m\left(\frac{x}{a} \pm \frac{Ht}{T}\right) = 0, \qquad \text{as consequence of } m = \infty$$

Je conviens que je ne me suis pas exprimé assex exactment ..

Today a mathematician would consider not completely rigorous only because a (easy) proof of the exchanges of limits needed to "pass to the continuum" is not even mentioned.

He returns on the matter looking only at motion interior to $[0, a]$: getts again Euler solution without the $\mathscr{C}_2$ condition: the method would be called today the search for a weak solution.

Key idea on the string has been applied in other cases: a most remarkable one is Quantum field theory. *i.e.* quantization of

$$\mathscr{L} = \frac{\mu}{2}\int_{\alpha}^{\beta}\Big(\dot{\varphi}(x)^2 - c^2(\frac{d\varphi}{dx}(x))^2 - (\frac{m_0 c^2}{\hbar})^2\varphi(x)^2 - I(\varphi(x))\Big)\,dx$$

$I(z) = \lambda z^4 + \mu z^2 + \nu$. If $I = 0$ is a vibrating string with density μ, tension $\tau = \mu c^2$ and an elastic pinning force $\mu(\frac{m_0 c^2}{\hbar})^2$.

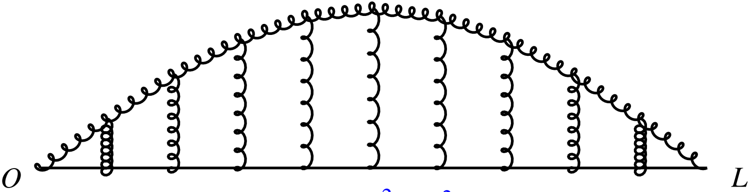


Fig.1

$$\text{“}\ (\mathscr{H}F)(\varphi) = \int_0^L\Big(-\frac{\hbar^2}{2\mu}\frac{\delta^2 F}{\delta\varphi(x)^2}(\varphi) + \frac{\mu}{2}\Big(c^2(\frac{\partial\varphi}{\partial\vec{x}}(\vec{x}))^2 + \\ +(\frac{m_0 c^2}{\hbar})^2\varphi(x)^2 + I(\varphi(\vec{x}))\Big)F(\varphi)\Big)\,d^D\vec{x}\ \text{”}$$

How to give a meaning? just as Lagrange; discretize

$$\mathscr{L} = \frac{\mu}{2}\delta \sum_{n\delta\in\Lambda_0} \Big(\dot{\varphi}_{n\delta}^2 - c^2 \sum_{j=1}^{D} (\varphi_{n\delta+e_j\delta} - \varphi_{n\varepsilon})^2 - (\frac{m_0 c^2}{\hbar})^2 \varphi_{n\varepsilon}^2 - I(\varphi_{n\varepsilon}) \Big)$$

with Hamiltonian

$$\mathscr{H}_\delta = -\frac{\hbar^2}{2\mu\delta} \sum_{n\delta\in\Lambda_o} \frac{\partial^2}{\partial\varphi_{n\delta}^2} + \frac{\mu\delta}{2} \sum_{n\delta\in\Lambda_o} \Big(c^2 \frac{(\varphi_{n\delta+e\delta} - \varphi_{n\delta})^2}{\delta^2} + (\frac{m_0 c^2}{\hbar})^2 \varphi_{n\delta}^2 + I(\varphi_{n\delta}) \Big)$$

At this point it could be claimed, with Lagrange,

Ces équations, comme il est aisé de le voir, sont en même nombre que les particules dont on cherche les mouvements; c'est pourqoi, le problème étant déjà absolument determiné par leur moyen, on est obligé de s'en tenir là, de sorte que toute condition étrangère ne peut pas manquer de rendre la solution insuffisante et même fautive.

Attempts to study $\mathscr{H}$ via expansions in λ, μ, ν immediately leads to *nonsensical results*. Even for φ^4 string or film cases, $d = 1, 2$)

In 1960's φ^4 fully understood by Nelson, Glimm-Jaffe, Wilson.

No infinities arise taking seriously discretized Hamiltonian computing physically relevant quantities and passing to the continuum limit, just as Lagrange did in his theory of sound.

This has been a major success of Physics and Analysis obtained through perturbation theory. Obtained by reduction to power series in a sequence of parameters $\Lambda_k \stackrel{def}{=} (\lambda_k, \mu_k, \nu_k)$, $k = 0,\ 1, 2, \ldots$, called *running coupling constants*, related by

$$\Lambda_k = M\Lambda_{k+1} + B(\{\Lambda_r\}_{r=k+1}^{\infty})$$

where M is a diagonal matrix with elements m_1, m_2, m_3. This is regrded as an equation for the sequence $(\Lambda_k)_{k=0}^{\infty}$ of which a solution is demanded bounded as $k \to \infty$.

The equation is an infinite dimensional version of what Lagrange calls a litteral equation: in the simplest case

$$\alpha = x - \varphi(x), \quad \longleftrightarrow \quad \psi(x) = \psi(\alpha) + \sum_{k=1}^{\infty} \frac{1}{k!} \partial_\alpha^{k-1} (\varphi(\alpha)^k \partial_\alpha \psi(\alpha))$$

for all ψ: a theorem by L. In particular for $\psi(\alpha) \equiv \alpha$

$$x(\alpha) = \alpha + \sum_{k=1}^{\infty} \frac{1}{k!} \partial_\alpha^{k-1} (\varphi(\alpha)^k), \qquad x(0) = \sum_{k=1}^{\infty} \frac{1}{k!} \partial_\alpha^{k-1} (\varphi(\alpha)^k)\Big|_{\alpha=0}$$

give inverse of $x + \varphi(x)$, resp., a solution of $x = \varphi(x)$. Widely used in celestial mechanics.

L. formula admits a combinatorial representation which has been used in QFT (above cases) and low temperature Physics, expressed in terms of tree graphs θ.

$\frac{1}{k!}\partial_\alpha^{k-1}(\varphi(\alpha)^k) = \sum_\theta \mathrm{Val}(\theta)$

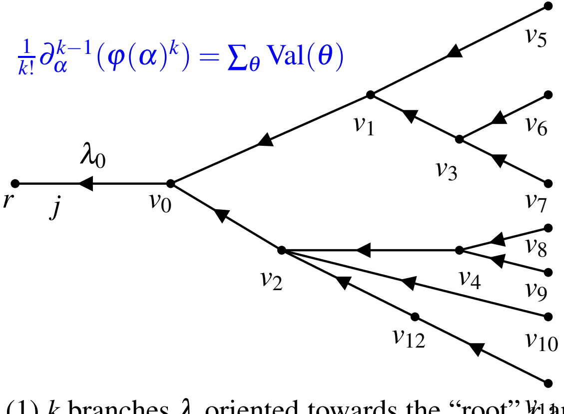


Fig.2

(1) k branches λ oriented towards the "root" r and

(2) at each node v enter k_v branches $\lambda_1 = vv_1, \ldots, \lambda_{k_v} = vv_{k_v}$

(3) each node carries a label $j_{v_1}, \ldots, j_{k_{j_v}}$ and

(4) the node v symbolizes the tensor $\partial^{k_v}_{x_{j_{v_1}}, \ldots x_{j_{v_{k_v}}}} \varphi(x)_{j_v}$

(5) Two decorated trees are equivalent "by pivoting"

(6) The *value*, $\mathrm{Val}(\theta)$ is defined as

$$\mathrm{Val}(\theta)=\prod_{v\in\theta}\Big(\frac{1}{k_v!}\partial^{k_v}_{j_{v_1},\ldots,j_{v_{k_v}}}\varphi_{j_v}(x)\Big)$$

where all indices except the $j=j_{v_0}$ associated with the root branch appear twice and summation over them is understood.

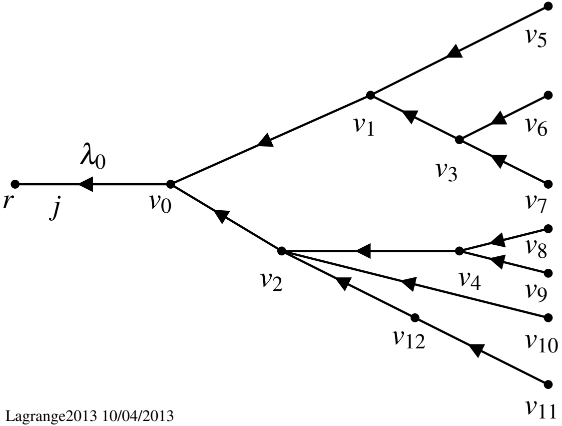


Fig.2

L. formula $\Rightarrow$ resum perturbative series: soon used for Kepler equation by Carlini (1817), Bessel (1817), Jacobi (1850), Levi-Civita (1909); recently for KAM theory, *i.e.* for

$$\vec{h}(\alpha)+(\omega\cdot\partial)^{-2}\varepsilon\,\partial f(\alpha+\vec{h}(\alpha)) \;=\; \vec{0}, \qquad \alpha\in T^{\ell}$$

solved in the same way writing $\vec{A}=\vec{h}+\mathscr{K}\vec{h}$ when $\vec{A}=\vec{0}$,

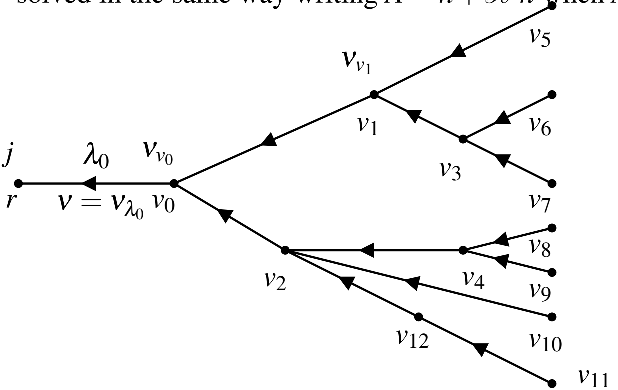


Fig.3

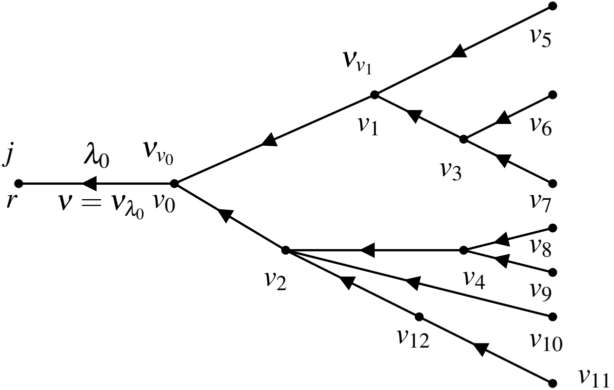


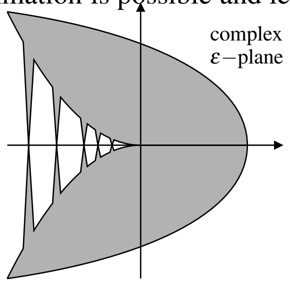
Fig.3

$$\vec{h}_\nu^{[k]} = \sum_\theta \mathrm{Val}(\theta), \qquad \mathrm{Val}(\theta) = \Big(\prod_v \frac{f_{\nu_v}}{k_v!}\Big)\Big(\prod_\lambda \frac{\nu_v \cdot \nu_{v'}}{(\omega\cdot\nu(\lambda))^2}\Big)$$

developed (in celestial mechanics) by Lindstedt and Newcomb. Proof of the KAM theorem ⇒ algebraic check in which the main difficulty of the *small divisors* becomes a combinatorial problem on cancellations (considered difficult): BUT:

D'ailleurs mes recherches n'ont rien de commun avec le leurs que le problème qui en fait l'object; et c'est toujours contribuer à l'avancement des Mathématiques que de montrer comment on peut résoudre les même questions et parvenir au même resultats par des voies très-différentes; les méthodes se prètent par ce moyen un jour mutuel et en acquièrent souvent un plus grand degré d'évidence et de généralité.

This led to study cases with defined-NOT(?) convergent series: resummation is possible and leads to analytic in ε in region like



contact points < 0
with Lebesgue density 1 at 0.

Or Borel summability in region
with $\varepsilon = 0$ on its boundary.

Most well known is *Traité de Mécanique Analytique*: reduction to first principles of every problem considered, constantly returning to applying the least action principle in the form of the combination the the principle of virtual works with the principle of D'Alembert.

The addition of the virtual works is a basic contribution of L. The book summarizes experience collected in studying

(1) Moon librations, perhaps first application of PT based on analytical mechanics

(2) rigid body, undertaken curiously avoiding starting from the proper axes of rotation; byproduct eigenvalues reality of 3×3 symm. matrix, and (later) integrability of "Lagrange's top"

(3) secular variations of the planetary elements and several other celestial questions (opening the way to Laplace)

(4) quadratures, of several dynamical systems

(5) The *Traité* also reflects the attitude of Lagrange regarding matter as constituted by particles: *i.e.* his consistently kept atomistic view. In one among several works on fluids:

Quoique nous ignorions la constitution intérieure des fluides, nous ne pouvons douter que les particules qui les composent ne soient matérielles, et que par cette raison les lois générales de l'équilibre ne leur conviennent comme aux corps solides,

(6) His Mechanics based on variational principle was adopted in developing further the atomistic theories: Clausius and Boltzmann make constant use of the action principle using also Lagrange's notations *e.g.* of the well known commutation rule $\delta d = d\delta$ introduced by L. and become universal.

(7) The style of all his papers (continued in the *Traité*) remarkably starts with a dense summary of others' previous works (an important help to historians)

(8) Very few notes written in Italian; a remarkable one has been inserted in the Tome VII of the collected works. A somewhat unusual algorithm to evaluate derivatives and integrals.

(9) Consideration of friction is not frequent in the works of L., it is mentioned marginally in a remark on the vibrating strings theory, the analysis on the tautochrone curves and on the variations of the planetary elements

(10) Although attentive to applications (for instance the best shape to give to a column) friction enters only very marginally in the remarkable theory of the anchor escapement: surprising.

(11) Infinitesimals, in the sense of Leibnitz, pervasive in his work (example is the mentioned letter (quoted above) showing that L. learnt very early ($<$1754) their use and their formidable power of easing the task of his future works long algebraic steps: this makes reading his papers easy and very pleasant

(12) However L. himself seems to have realized that something ought to be done in systematizing the foundations of analysis: his lecture notes

Théorie des Fonctions analytiques, contenants les principes du calcul differentiel dégagé de toutes considérations d'infiniment petits ou d'evanouissants, de limites ou de fluxions et réduits à l'analyse algébrique de quantités finies

and the very first page of the

Lecons sur le calcul des fonctions

are a clear sign of his hidden qualms on the matter.

(13) An informative history of his life can be found in the "Notices sur la vie et les ouvrages" in the Preface by M. Delambre of Tome I of the *Oeuvres* and in the Elogio by P. Cossali at the University of Padova.

References: see the distributed notes