Renormalization group, Kondo effect and hierarchical models

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1-d lattice, fermions+impurity, "Kondo problem"

$$H_h = \sum_{\alpha=\pm} \Big(\sum_{x=-L/2}^{L/2-1} \psi^+_\alpha(x)\,(-\frac{1}{2}\Delta - 1)\,\psi^-_\alpha(x) + h\ \varphi^+ \sigma^z \varphi^- \Big)$$

$$H_K = H_0 + \lambda \sum_{\substack{\alpha,\alpha'=\pm \\ \gamma,\gamma'=\pm}} \sum_{j=1}^{3} \psi^+_\alpha(0)\sigma^j_{\alpha,\alpha'}\psi^-_{\alpha'}(0)\ \varphi^+_\gamma \sigma^j_{\gamma,\gamma'} \varphi^-_{\gamma'} = H_h + V$$

(1) $\psi^\pm_\alpha(x), \varphi^\pm_\gamma$ C&A operators, σ^j, $j = 1, 2, 3$, Pauli matrices

(2) $x \in$ unit lattice, $-L/2$, $L/2$ identified (periodic b.c.)

(3) $\Delta f(x) = f(x+1) - 2f(x) + f(x-1)$ discrete Laplacian.

If $\lambda = 0$ impurity-electrons independent: classic or quantum

$$\chi(\beta,h) \propto \beta \xrightarrow[\beta\to\infty]{} \infty, \qquad \forall\ L \geq 1,\ \beta h < 1$$

Interaction (classic) elec.+imp.: field on both & $\lambda \neq 0$

$$\chi(\beta,h) = 4\beta\frac{(1+e^{-2\lambda\beta}\cosh\beta h)}{(\cosh 2\beta h + e^{-2\lambda\beta})^2} \xrightarrow[\beta\to\pm\infty]{} \begin{array}{r l} 0 & \text{repulsive} \\ +\infty & \text{attractive} \end{array}$$

field on impurity only: $\chi(\beta,0) = \beta \to \infty$
Reason: $\lambda < 0$ $\to$ rigidly antiparallel spins **????**

Still true if $L < \infty$ classic&quantum or $L = \infty$ classic

XY model confirms (∞ both cases, exact)

Then Trivial? (0 repulsive, ∞ attractive ?)

BUT

If $L = \infty$ quantum chain: new phenomena

1) at $\lambda = 0 \Rightarrow$ Pauli paramagnetism (1926)

local or specific suscpt. $< \infty$ at $T \geq 0$:

$$\chi(\infty, 0) = \rho \frac{1}{k_B T_F} \frac{d}{2}, \qquad (Pauli)$$

2) at fixed $\lambda < 0 \Rightarrow$ Kondo effect:

susceptibility $\chi(\beta, h)$

smooth at $T = 0$ and $h \geq 0$

Kondo realized the problem (3^d-order P.T.) and gave arguments (1964) for $\chi < \infty$ (actually conductivity $< \infty$)

Anderson-Yuval-Hamann (1969,70) $\Rightarrow$ multiscale nature of the problem, relation with the 1D Coulomb gas & solved the $\lambda > 0$ case (no Kondo eff.), & stressed that lack of asymptotic freedom = obstacle for $\lambda < 0$

Wilson (1974-75) overcame asymptotic freedom by discussing a somewhat modified model and finding a recursion scheme, numerically implementable in an appropriately simplified model.

The method built a sequence of approximate Hamiltonians (with finitely many coefficients) more and more accurately representing the system on larger and larger scales, leading to the Kondo effect via a nontrivial fixed point.

Evaluate $Z = \mathrm{Tr}\, e^{-\beta H_K}$ as a functional integral, (BG990).
The free fields $\psi^\pm(x), \varphi^\pm$

$$\psi^\pm(x) = \sum_m e^{\pm ikx} \psi^{\pm[m]}(x),\;\; \varphi^\pm = \sum_m \varphi^{\pm[m]}$$

can be decomposed into components of scale 2^{-m}, $m \in Z$

$$\psi^\pm(x) = \sum_{m=0}^{-\infty} \sum_{\omega=\pm} e^{\pm i\omega p_f x}\, 2^{\frac{1}{2}m}\, \psi_\omega^{\pm[m]}(2^m x), \quad \varphi^\pm = \sum_{m=0}^{-\infty} \varphi^{\pm[m]}$$

quasi particles, neglecting the UV (*i.e.* $m \le 0$). Then represent Z as a Grassmann integral.
Fields become Grassman variables.

But since the impurity is localized observ. localized at 0 depend on fields at 0, $\psi^\pm(0), \varphi^\pm \Rightarrow$ **1D problem** (AYH).

Key: response to field h acting on impurity site only depends on the propagators with $x=0$.

By Wick $\Rightarrow$ only average values, over "time" of propagators at $x=0$ needed. Propagators on scale m are $g^{[m]}(t-t')$

$$\delta_{m,m'}\sum_{\omega}\int\frac{dk_0dk}{(2\pi)^2}\frac{e^{ik_0(t-t')}}{-ik_0+\omega e(k)}\chi(2^{-2m}(k_0^2+k^2)),$$

$$\delta_{m,m'}\int\frac{dk_0}{2\pi}\frac{e^{i\sigma k_0(t-t')}}{-i\sigma k_0}\chi(2^{-m}\frac{k_0}{2\pi})$$

singularity at $t-t'=0$ (UV sing.) and at $t-t'=\infty$ (IR sing.) regularized via χ on scale 2^{-m}; $e(k)=-\cos k$.

Illustration of (AYH970) remark: 1D problem, (long range)

$$\text{Main operators}:\quad \vec{A}_x \stackrel{def}{=} \psi^+_x\boldsymbol{\sigma}\psi^-_x,\ \vec{B}_x\stackrel{def}{=}\varphi^+\boldsymbol{\sigma}\varphi^-$$

Interaction Ham. is constructed via the operators

$$O_0 = -\lambda^0 \vec{A}\cdot\vec{B},\ O_1 = \lambda^1 \vec{A}^2,\ O_2 = \lambda^2 \vec{B}^2,\ O_3 = \lambda^3 \vec{A}^2 \vec{B}^2$$

H_K on scale $m = 0$ is (with $\lambda^0 < 0$ and $\lambda^1 = \lambda^2 = \lambda^3 = 0$)

$$H_K = H_0 - \sum_x (\lambda^0 O_{x,0} + \lambda^1 O_{x,1} + \lambda^2 O_2 + \lambda^3 O_{x,3}) + \dots$$

Set RG analysis via (Grassmannian) as BG990 for $\mathrm{Tr} e^{-\beta H_K}$

Scaling O_0 = marginal, O_2 = relevant

Difficulty is immediate: multiscale PT at $h = 0$ generates a power series with at least the above 4 running costants $(\boldsymbol{\lambda}_n)$ $n \le 0$. Should be related by recurrence

$$\boldsymbol{\lambda}_n = \Lambda \boldsymbol{\lambda}_{n+1} + \mathcal{B}(\boldsymbol{\lambda}_{n+1}), \quad \lambda_0 = (-\lambda, 0, 0, 0)$$

with $\Lambda = (1, \frac{1}{2}, 2, \frac{1}{2})$ and $\mathcal{B}$ is a formal series.

Even forgetting convergence, PT of no use: marginal term grows (if $\lambda_0 < 0$) and generates relevant term!

To understand a simpler problem turn to hierarchical model

The propagators $g^{[m]}(t-t')$ are c̃onstant for $t > t'$ on scale m, *i.e.* $t, t' \in I_m = [n2^{-m}, (n+1)2^{-m}]$, antisymmetric in t, t' and fast decay on scale 2^{-m}

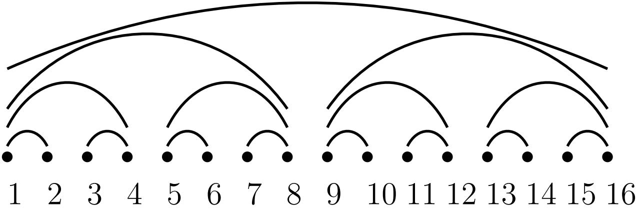
Hierarchical fields will be defined by assigning to each I_m two Grassmannians $2^{\frac{1}{2}m} z^{[m]}(t), \zeta^{[m]}(t)$

1) exactly constant in each half of I_m

2) propagator 1 for $t \in I_m^-, t' \in I_m^+$, -1 for $t \in I_m^+, t' \in I_m^-$

3) independent for $t \in I_m$, $t' \in I_{m'} \neq I_m$

$$\psi_\alpha^{[\le m]\pm}(t) = 2^{\frac{m}{2}} \Big(z_\alpha^{[m]\pm}(t) + \frac{1}{\sqrt{2}} Z_\alpha^{[m-1]\pm} \Big),$$
$$\varphi_\beta^{[\le m]\pm}(t) = \zeta_\beta^{[m]\pm}(t) + \Xi_\beta^{[m-1]\pm}$$



Hierarchy of lattice sites $[1, \ldots, 2^N]$: i intervals on scale 0

$$\psi_\alpha^{[\le m]\pm}(t) = 2^{\frac{m}{2}}\left(z_\alpha^{[m]\pm}(t) + \frac{1}{\sqrt{2}} Z_\alpha^{[m-1]\pm}\right),$$

$$\varphi_\beta^{[\le m]\pm}(t) = \zeta_\beta^{[m]\pm}(t) + \Xi_\beta^{[m-1]\pm}$$

where z, ζ are fields of scale m while Z e Ξ are constant on scale m (not $m-1$).