Friction & reversibility in the Lorenz96 equation and chaotic hypothesis (V.Lucarini & GG)

Stationary states: $\Rightarrow$ probab. distrib. on phase space.

Collections of stationary states $\Rightarrow$ ensembles $\mathcal{E}$: in equilibrium give the statistics (canonical, microc., &tc).

Can this be done for stationary nonequilibrium? Motion:

$$\dot{x}_j = f_j(x) + F_j - \nu\,(Lx)_j, \qquad \nu > 0,\ j = 1, \ldots, N$$

$L > 0$ dissipation matrix: e.g. $(Lx)_j = x_j$, $\nu > 0$ (friction), $f(x) = f(-x)$ (time reversal)

Chaotic hypothesis: "think of it as an Anosov system" (Cohen,G)

(analogue of the periodicity≡ergodicity hypothesis of Boltzmann, Clausius, Maxwell, and possibly as unintuitive)

Time reversal symmetry is violated by friction.

BUT it is a fundamental symmetry: $\Rightarrow$ possible to restore?

How? in which sense? Start from a special case:

the Lorenz96 eq. (periodic b.c.)

$$\dot{x}_j = x_{j-1}(x_{j+1} - x_{j-2}) + F - \nu x_j, \qquad j = 0, \dots, N-1$$

Vary ν and let μ_ν stationary distrib. Let $\overline{E} = \langle \sum_j x_i^2 \rangle_{\mu_\nu}$:
this is an "ensemble" (viscosity ensemble)

$$\text{Replace } \nu \text{ by} \qquad \alpha(x) = \frac{\sum_i F x_i}{\sum_i x_i^2}$$

New Eq. has $E(x) = \sum_i x_i^2$ as exact constant of motion

$$\dot{x}_j = x_{j-1}(x_{j+1} - x_{j-2}) + F - \alpha(x) x_j,$$

Vary E and let μ_E station. distrib.: (viscosity ensemble)
Volume contracts by $\sum \partial_j (a(x) x_j)$

$$\sigma(x) = (N-1)\alpha(x), \quad p = \tau^{-1} \int_0^\tau \sigma(x(t)) dt / \langle \sigma \rangle$$

Conjecture: Equivalent ensembles:

State $\widetilde{\mu}_E$ labeled by E corresponds to states μ_ν labeled by ν are equivalent if $\widetilde{\mu}_E(\alpha(x)) = \mu_\nu(E(x))$

$$\mu_\nu \sim \widetilde{\mu}_E \longleftrightarrow E = \mu_\nu(E(\cdot)) \longleftrightarrow \nu = \widetilde{\mu}_E(\alpha(\cdot))$$

Give the same statistics in the limit of large $R = \frac{F}{\nu^2}$.

Analogy canonical μ_β = microcanonical $\widetilde{\mu}_E$ if

$$\mu_\beta(E(.)) = E \longleftrightarrow \widetilde{\mu}_E(K(.)) = \frac{3}{2\beta}N$$

in the limit of large volume (fixed density or specific E).

Why? several reasons. Eg. chaoticity implies

$$\alpha(x(t)) = \frac{\sum_i F x_i}{\sum_i x_i^2} \qquad \text{“self − averaging”}$$

Tests performed at $N = 32$ (with checks up to $N = 512$) and high R (at $R > 8$, system is very chaotic with > 20 Lyap.s exponents and at larger R it has $\sim \frac{1}{2}N$ L.e.)

1) $\mu_{\overline{E}}(\alpha) = \nu \longleftrightarrow \mu_\nu(E) = \overline{E}$

2) If g is reasonable ("local") observable $\frac{1}{T}\int_0^T g(S_t x)dt$ has same statistics in both

3) The "Fluctuation Relation" holds for the fluctuations of phase space vol (reversible case): reflect chaotic hypothesis

4) Found its N-independence and ensemble independence of the Lyapunov spectrum (Livi,Politi,Ruffo)

5) In so doing found or confirmed several scaling and pairing rules for Lyapunov exponents (somewhat surprising) and checked a local version of the F.R.

Scaling of energy-momentum (irreversible model):

$$E = \sum_i x_i^2, \qquad M = \sum_i x_i$$

$$\frac{\overline{E}^i_R}{N} \sim c_E R^{4/3}, \quad \frac{\overline{M}^i_R}{N} \sim 2c_E R^{1/3} \quad c_E = 0.59 \pm 0.01$$

$$\frac{std(E)^i_R}{N} = \frac{\left(\overline{E^2}^i_R - (\overline{E}^i_R)^2\right)^{1/2}}{N} = \tilde{c}_E R^{4/3}, \quad \tilde{c}_E \sim 0.2 c_E$$

$$\frac{std(M)^i_R}{N} = \tilde{c}_M R^{2/3} \quad \tilde{c}_E \sim 0.046 \pm 0.001$$

$$t^{i,M}_{dec} \sim c_M R^{-2/3} \quad c_M = 1.28 \pm 0.01$$

The first two confirm Lorenz96, the 3d,4th "new", 5th is the "decorrelation" time $\langle\, M(t) M(0)\,\rangle$

Black: Lyap. exp.s $R = 2048$
Magenta: $\pi(j) = (\lambda_j + \lambda_{N-j+1})/2$.
Blue: Lyap. exp.s $R = 256$
value of $\pi(j)$ at $R = 252$ (invisible below magenta).

Pairing accuracy. Irreversible model.

Blue: $\pi(j) = (\lambda_j + \lambda_{N-j+1})/2$, $8 < R < 2048, N = 32$.

Almost constant: as it can be seen if compared to λ_j. The small variation reflects the fact that the spectrum shows an asymptotic shape.

Pairing accuracy. Irreversible model.

Blue: $\pi(j) = (\lambda_j + \lambda_{N-j+1})/2$, $8 \leq R \leq 2048, N = 32$.

Almost constant: as it can be seen if compared to λ_j. The small variation reflects the fact that the spectrum shows an asymptotic shape.

Continuous limit of Lyapunov Spectrum (LPR):
asymptotics in $N = 32, 256$ at R fixed:

$R = 256$: λ_j for $N = 256$ and Black mark $N = 32$
red line $\pi(j) = (\lambda_j + \lambda_{N-j+1})/2$
for $N = 256$ and marker for $N = 32$; zoom

Scaling Lyapunov Spectrum: $8 \leq R = 2^n \leq 2048$

$$x = \frac{j}{N+1} \;\Rightarrow\; |\lambda(x) + \pi(x)| \sim c_\lambda \, |2x - 1|^{5/3} \, R^{2/3}$$
$$\sim |\lambda(x) + 1| \sim c_\lambda \, |2x - 1|^{5/3} \, R^{2/3}, \quad c_\lambda \sim 0.8$$

Blue: $|\lambda_j + 1|/(c_\lambda R^{2/3})$, Black: $|2j/(N+1) - 1|^{5/3}$

Dimension of Attractor

The $|\lambda(x)+1| \sim c_\lambda \, |2x-1|^{5/3} \, R^{2/3}$ yields the full spectrum: hence can compute the KY dimension

$$N - d_{KY} = \frac{N}{1 + c_\lambda R^{\frac{2}{3}}} \xrightarrow[R\to\infty]{} 0, \qquad \forall\ N$$

attractor has a dimension virtually indistinguishable from that of the full phase space.

However SRB distribution deeply different from equidistribution (often confused with ergodicity): made clear by the equivalence (if holding) and the validity of the Fluctuation Relation needs test

Reversible-Irreversible ensembles equivalence:

Black: pdf for M/N rev, $R = 2048$. Blue $-$ pdf for M/N irrev for $R = 2048$. Red black + blue line. Note vertical scales.

Check Fluctuation Relation (FR)

$$p=\frac{1}{\tau}\frac{\int_0^\tau \sigma(x(t))dt}{\langle\,\sigma\,\rangle_{srb}}$$

$$\frac{1}{\tau\overline{\sigma}_R}\log\frac{P_\tau^R(p)}{P_\tau^R(-p)}=p \quad ???$$

F.R. slope $c(\tau)\xrightarrow[R\to\infty]{}1,\ R=512$

$$c(\tau)=1+\left(\frac{t^{r,\sigma}_{dec,R}}{\tau}\right)^{4/3}=1+\left(\frac{c_\sigma}{\tau}\right)^{4/3}R^{-8/9}$$

Check Fluctuation Relation

F.R. $R = 2048$, approach 1 as $\tau \uparrow$ beyond decorrelation time

Local F.R. for $R = 2048$

$$\frac{1}{\tau}\log\frac{P_\tau^R(p)}{P_\tau^R(-p)} = \overline{\sigma^\beta}_R p + O(\tau^{-1}) = \beta\overline{\sigma}_R p + O(\tau^{-1})$$

Lyapunov exp. in reversible casereversible $\equiv$ irrev

Same picture as the irreversible case: graphs overlap

Red: Lyap exps $R = 2048$. Magenta $(\lambda_j + \lambda_{N-j+1})/2$. Blue Lyaps $R = 2048$. Black: $(\lambda_j + \lambda_{N-j+1})/2$

Consequently Reversible pairing occurs.

Equivalent Ensembles (more) general theory

$E(x)$ observable s.t. $\sum_{j=1}^{N} \partial_j E(x)(Lx)_j = M(x) > 0 \; x \neq 0$.
E.g. $L = 1,\; E(x) = \frac{1}{2}\sum_j x_j^2, \Rightarrow M(x) = x^2$.

$$\dot{x}_j = f_j(x) + F_j - \nu(Lx)_j, \qquad \nu > 0,\; j = 1, \ldots, N$$

$$\dot{x}_j = f_j(x) + F_j - \alpha(x)(Lx)_j, \qquad \alpha(x) \stackrel{def}{=} \frac{\sum_{j=1}^{N} F_j \partial_j E}{M(x)}$$

Dissipation balanced on $E(x) \Rightarrow E(x(t)) = const$

Define $\mathcal{E}$ and $\widetilde{\mathcal{E}}$: conjectured is equivalence at large forcing (when both satisfy Chaotic hypothesis for $\langle\, \alpha(x(t))\alpha(x(0))\,\rangle$ is finite).

Lorenz96 is one example
Other examples: NS equation (periodic container $\mathcal{O}$)

with viscosity ν

$$\dot{\vec{u}} + (\vec{u}\cdot\boldsymbol{\partial})\vec{u} = -\boldsymbol{\partial} p + \vec{g} + \nu\Delta\vec{u} = 0, \qquad \boldsymbol{\partial}\cdot\vec{u} = 0$$

and equivalent eq. balanced on the "dissipation" observable $E(\vec{u}) = \int_{\mathcal{O}} (\partial\vec{u}(x))^2 dx$

$$\dot{\vec{u}} + (\vec{u}\cdot\boldsymbol{\partial})\vec{u} = -\boldsymbol{\partial} p + \vec{g} + \alpha(\vec{u})\Delta\vec{u}, \qquad \boldsymbol{\partial}\cdot\vec{u} = 0$$

$$\alpha(\vec{u}) \stackrel{def}{=} \frac{\sum_{\vec{k}} \vec{k}^2\, \vec{g}_{\vec{k}}\cdot\vec{u}_{-\vec{k}}}{\sum_{\vec{k}} \vec{k}^4 |\vec{u}_{\vec{k}}|^2}, \qquad D = 2$$

If $D = 3$ similar expression (more involved because vorticity is not conserved in inviscid case)

$N = 168$: $R^2 = 10^6$; viscous NS $(+)$,energy $(*)$, enstr $(\boxdot)$

Lyap exps $N = 168$: $R^2 = 10^6$, force on $\pm(4,-3), \pm(3,-4)$

viscous $(+)$ at force on $\pm(4,-3), \pm(3,-4)$
$(\times) = (\lambda_k + \lambda'_k)/2$
energy $(*)$
enstrophy $(\boxdot)$, or
palinstrophy $(\blacksquare)$.

Runs lengths $T \in [125, 250]$, units of $1/\lambda_{max}$, λ_{max}.

Error bars identified with symbols size.

Overlap of the 4 spectra (approximate, because of numerical fluctuations in quantities that should be exact constants)

NS too $\Rightarrow$ hints at extending equivalence to spectra.

References

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