Comments on Ruelle's analysis of the OK41 turbulence theory (P. Garrido & GG)

Brief survey of OK41 (Obukov-Kolmogorov) on N-S

$$\dot{\underline{u}}_{\underline{k}} = -\, i \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \underline{u}_{\,\underline{k}_1} \cdot \underline{k}_{\,2} \Pi_{\underline{k}} \underline{u}_{\underline{k}_2} + \underline{g}_{\underline{k}} \delta_{|\underline{k}|=k_0} \qquad k_0 \le |\underline{k}| \le k_\nu$$

$$\dot{\underline{u}}_{\underline{k}} = -\, \nu \underline{k}^2 \underline{u}_{\underline{k}} - i \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \underline{u}_{\,\underline{k}_1} \cdot \underline{k}_{\,2} \Pi_{\underline{k}} \underline{u}_{\underline{k}_2} \qquad k_\nu < |\underline{k}| < k'_\nu$$

Scale length (k_0^{-1}) "injection scale"

Scale length $\ge k_\nu^{-1}$ "inertial range: ν negligible, turbulence

Scale length in $[{k'}_\nu^{-1}, k_\nu^{-1}]$ "dissipation": laminar motion

To determine k_ν ? Define

Energy variation rate in modes at scales $< \sigma^{-1}$:

$$E^\sigma \overset{def}{=} \frac{1}{2}\frac{d}{dt} L^3 \sum_{|\underline{k}|<\sigma^{-1}} |\underline{u}_{\underline{k}}|^2 = \mathcal{L} - \mathcal{E}_{\sigma,\sigma'} - \mathcal{E}_{\sigma,\sigma',\infty} + 0$$

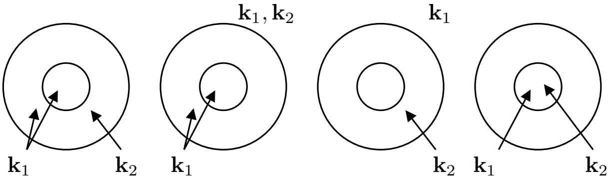
$\mathcal{L}$=forcing work (on scale k_0^{-1}); $2L^3 \mathrm{Re}\,\overline{\underline{g}}_{\underline{k}_0} \cdot \underline{u}_{\underline{k}_0}$

$\mathcal{E}_{\sigma,\sigma'}$= work on modes $|\underline{k}| < \sigma$ due to $|\underline{k}| \in [\sigma, \sigma')$:

$$\mathcal{E}_{\sigma,\sigma'} = iL^3 \sum_{|\underline{k}_3|<\sigma} \Big(\sum_{\substack{\underline{k}_1+\underline{k}_2+\underline{k}_3=\underline{0} \\ |\underline{k}_1|\in[k_0,\sigma'),\ |\underline{k}_2|\in[\sigma,\sigma']}} \underline{u}_{\underline{k}_1} \cdot \underline{k}_2\, \underline{u}_{\underline{k}_2} \Big) \cdot \underline{u}_{\underline{k}_3}$$

$\mathcal{E}_{\sigma,\sigma',\infty}$=work "modes $|\underline{k}| < \sigma$ from $|\underline{k}_2| \in [\sigma', \infty)$ or $|\underline{k}_2| \in [\sigma, \sigma'), |\underline{k}_1| \in [\sigma', \infty)$

0=internal work inside$|\underline{k}| < \sigma$



$\mathcal{E}_{\sigma,\sigma'}$ first, $\mathcal{E}_{\sigma,\sigma',\infty}$ second and third, 0 fourth.

The second and third involve pairs of $\underline{u}_{\underline{k}}$ separated by the gap.

0) **Homogeneous turbulence**: $\langle\, \mathcal{E}_{\sigma,\sigma',\infty} \,\rangle = 0$ *if* σ *and* $\sigma' = \kappa\sigma$ *are* $< k_\nu$*, in asymptotic regime.*

Provided κ *large enough*: to neglect exchanges btwn noncontiguous scales. Hence $\mathcal{E}_{\sigma,\sigma'} = L^3\varepsilon$ constant.

Energy dissipated only on scales $|\underline{k}| > k_\nu$, "cascading" without dissipation from large scales to small ones.

ε can only depend on velocity v_σ characteristic of lenfgth scale σ and on σ: hence $\varepsilon = \frac{v_\sigma^3}{\sigma}$

$$v_\sigma^2 \stackrel{def}{=} \langle \Big(\frac{1}{\Delta_\sigma}\int_{\Delta_\sigma} (\underline{u}(\underline{x}) - \underline{u}(\underline{x}_0))d\underline{x}\Big)^2 \rangle$$

2) $\underline{k}$-th FT of velocity var. assumed statistically indep,

3) Velocity var. in same scale boxes assumed stat. indep.

Then OK41 $\Rightarrow$ energy is $K(\sigma)d\sigma = const\, \varepsilon^{\frac{2}{3}}\sigma^{-\frac{5}{3}}$

Reynolds num. on scale σ: $R_\sigma = \frac{v_\sigma \sigma}{\nu}$ is (as $\frac{v_\sigma^3}{\sigma} = const$!)

$$R_\sigma = \frac{v_\sigma \sigma}{\nu} = \frac{v_L L}{\nu}\big(\frac{\sigma}{L}\big)^{\frac{4}{3}}$$

The Kolmogorv scale is then defined by the scale at which motion is laminar, *i.e.* $R_\sigma = 1$: this is

$$k_\nu = \sigma_\nu^{-1} = L^{-1}R^{\frac{3}{4}}$$

Dimension of attractor $\propto R^{\frac{9}{4}}$

As $\frac{v_\sigma^3}{\sigma} = \varepsilon \Rightarrow \tau_p = 0$

$$\langle v_\sigma^p \rangle = \varepsilon \, \sigma^{\frac{p}{3}(1+\tau_p)} \equiv \varepsilon \, \kappa^{-\frac{n}{3}(1+\tau_p)}, \qquad \sigma = L\kappa^{-n}$$

Of course v_σ fluct., observably, $\Rightarrow \tau_p \not\equiv 0$: except for $p = 1$.

Why? Ruelle, [3, 4] :
0) v_σ^3/σ be a r.v. with constant average on scale $\sigma = \kappa^{-n}$

1) In a box Δ_σ there are κ^3 boxes of scales $\frac{\sigma}{\kappa}$

$$\int v_n^3 P(v_n|v_{n-1}) dv_n \equiv \kappa^{-1} \, v_{n-1}^3, \qquad \text{constraint}$$

2) which one? "Boltzmannian": *i.e.* maximizing entropy under constraint $\int (-\log P(v|w) - \lambda v^3) P(v|w) dv = \max \Rightarrow$

$$W_n \stackrel{def}{=} v_n^3, \qquad P(W_n|W_{n-1}) = e^{-\frac{\kappa W_n}{W_{n-1}}} \frac{\kappa dW_n}{W_{n-1}}$$

$$\mathcal{P}_r(d\underline{W}) = \prod_{m=0}^{n} \prod_{i=1}^{\kappa^m} \frac{dW_{i,m+1}}{\kappa W_{i'm}} \, e^{-\frac{W_{i,m+1}}{\kappa W_{i',m}}}$$

Predictions? if $\sigma_n \stackrel{def}{=} \kappa^{-n}$

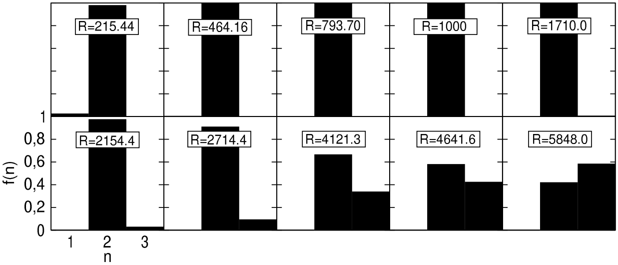
$$\langle\, v_n^p \,\rangle = \kappa^{-\frac{np}{3}} \kappa^{-n\tau_p}, \;\; \tau_p = -\frac{\log \Gamma(1+p)}{\log \kappa}$$

One free parameter: κ. Experimental data $\exists$ for $p < 18$:

Fit gives $\kappa \sim 22$, "very large" (!?).

Surprising: $\Rightarrow$ at "moderate" R's the number of scales to reach K-scale can be calculated and is at most 2 (*i.e.* $3, 4$ exist but are very rare).

Recall that this all started from the new finding (Schumacher et al.) that universality starts being manifest already at small Re.



Distribution of events that reach the Kolmogorov scale κ^{-n} for different values of the Reynold's numbers R and $\kappa = 22$. The total number of events is 10^{12}.

Another quantity measured (simulations) is "radial" velocity component distribution at Kolmogorov scale (or at any inertial scale where Re goes below a prefixed treshold):

$$n(\underline{W}) = \text{first } i \text{ s.t.} R_i = \frac{W_i^{\frac{1}{3}}}{\kappa^{-i}} \equiv \frac{v_i}{\kappa^{-i}} < 1$$

κ^{-i} is the K. scale. Ask probability dens. that at the K. scale radial velocity $\frac{v_i \cos\theta}{\langle v_i^2 \cos^2\theta \rangle^{\frac{1}{2}}} \in [z, z+dz]$

$$p(z) = \frac{1}{2}\mu_2^{\frac{1}{2}} P(\mu_2|z|)$$

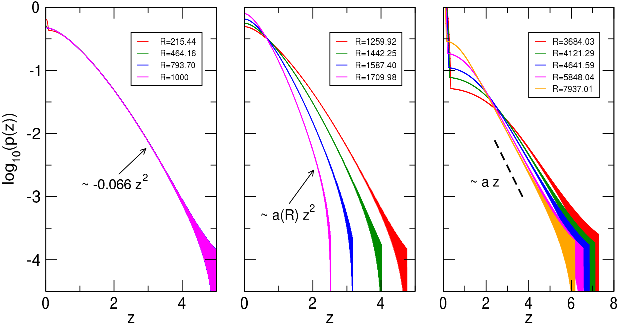
In (Schumacher et al, 2014) it was found

1) the $p(z)$ is Gaussiann at low Re.

2) but at moderate Re. $\log p(z)$ develops a linear tail

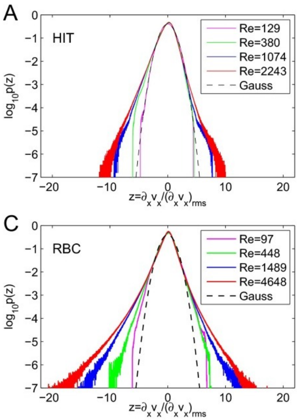
3) achieved via impressive simulations

Ruelle's distribution tests the above result

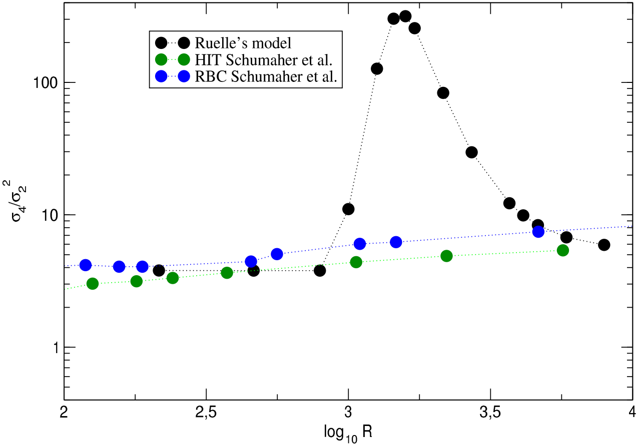


$\log_{10} p(z)$ distribution for different Reynold's numbers.
Center: $a(1259.92) = -0.08$, $a(1442.25) = -0.12$, $a(1587.40) = -0.24$, $a(1709.98) = -0.52$.
Right: $a(3684.03) = -0.51$, $a(4121.29) = -0.55$, $a(4621.59) = -0.58$,

$$a(5848.04) = -0.61, a(7937.01) = -0.64.$$



$p(z)$ by Schumaher et al. for varying Re.'s



Flatness (σ_4/σ_2^2) compared to Schumaher et al. [2] varying Re.'s

Computation of $P^*(\xi)$ is a problem on extreme events

$$W_n^{\frac{1}{3}}\kappa^n \equiv W_0^{\frac{1}{3}} w_1 w_2 \dots w_n, \qquad w_k = \frac{\kappa^k W_k}{\kappa^{k-1} W_{k-1}}$$

$$P_n(dw_1, \dots dw_n) = \prod_{i=1}^{n} p(w_i) dw_i$$

for i.d. random variables; extreme event, $*$, is

$$\frac{\log W_0}{3} + \varphi \sum_{i=1}^{m} \log w_i > 0 \ \forall m < n$$

$$\&\& \qquad \frac{\log W_0}{3} + \sum_{i=1}^{n} \log w_i < 0$$

with probability:

$$P^*(\xi) = \sum_{n=1}^{\infty} \int^{*} P_n(d\underline{w}) \delta(\sum_{i=1}^{n} \log w_i - \xi)$$

If $p(w)$ is not faster than exp., as in R, analysis of P^* involves the Gumbel distribution $\Phi(t) = e^{3t - e^{3t}}$

It should also be true **(?)** that if values of n really matter (*i.e.* at much larger *Re.*)

no matter which $p(w)$ is used
provided with exponential tail

the result will be the same as with R's $p(w) = e^{-w}dw$.

If so the *"Boltzmannian prescription"* would be set in a conceptually general perspective,

And the universality of the tails of the dissipation-pdf, *i.e.* pdf of $\frac{v_\sigma^3}{\sigma}$ in the inertial range may be perhaps more clear.

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