SRB & Boltzmann's distributions: hyperbolicity, ergodic hypothesis, ensembles

Basic assumption ("chaotic hypothesis") refers to equilibrium and nonequilibrium phenomena alike: once a system undergoes chaotic motions it shows them in a maximal sense: $\Rightarrow$ to be intended, for the purpose of modeling its behavior, an "Anosov system", [1].

Trivial to exhibit counterexamples: *e.g.*the horocyclic flow on a surface of constant < 0 curvature, [2].

However keep in mind that Statistical Mechanics developed from (Boltzmann, Clausius, Maxwell) supposing periodicity of the microscopic motions, [3, 4, 5, 6],

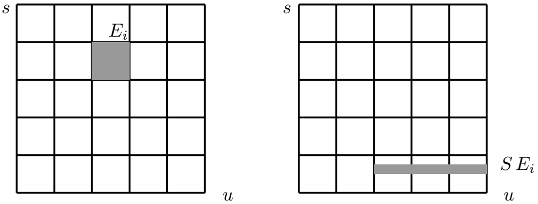
Chaotic system admits (uncountably) many invariant distributions on their microscopic configurations; how do the ones relevant for Physics emerge?, [7, 8, 9] Boltzmann realized the problem in 1871, [10], casting doubts on his early derivation (1868, [11]) of the microc. ensemble; resolved them essentially via ergodic hypothesis.

It became soon dubious whether phase space volume invariance (Hamiltonian) is of any help: many attempts to connect micr. evolution with macr. properties via "coarse grained" representations have met difficulties (hyperbolic nature $\Rightarrow$ strong phase space cells in deformation).

In stationary states out of equilibrium volume invariance not even true. And the many invariant distr. are all (strongly) mixing $\Rightarrow$ difficult to see how the Boltzmann or SRB distr. could emerge as the ones relevant for Physics, [7, 8].

Physics: hyperbolicity, (restrict to maps for simplicity and assume chaotic hypothesis), is essential to establish a coherent notion of "coarse grain" descriptions, [12, 13, 6].

Let (M, S) : be a smooth Anosov map S on the phase space M and let $\mathcal{E} = (E_1, \ldots, E_n)$ be a "Markov partition" of M



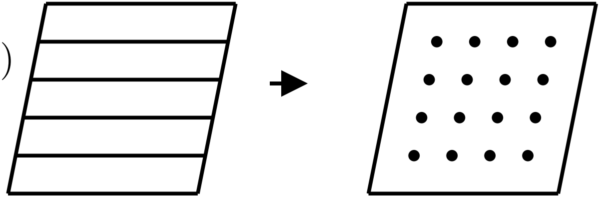
A $E_i \in \mathcal{E}$ transformed into SE_i s,t, shrinking part of boundary ends up exactly on boundary of some among $E_1, E_2, \ldots, E_n$. $\mathcal{E}$ chosen "**fine enough**" for the few interesting observables to be constant in each cell E_i.

$\Rightarrow$ symbolic represent. of motion: points $\rightarrow$ symbols $\boldsymbol{\sigma} = (\sigma_i)_{-\infty}^{\infty}$ with $M_{\sigma_i, \sigma_{i+1}} = 1$ determined by "transition matrix" $M_{\sigma,\sigma'} = 0, 1$ (transitive, *i.e.* $\exists n : (M^n)_{\sigma,\sigma'} > 0$).

"Ergodic hypothesis", classical, phase space discretized on regular lattice $R^{3N}_{\delta q} \times R^{3N}_{\delta p}$, of meshes $\delta q, \delta p$, and in confined Ham. sys. motion $\longleftrightarrow$ one cycle permutation of the (finite) set of points on given energy surface.

But conservative motions not required although, in general, *e.g.*out of equilibrium, evolution cannot be cyclic permutation. Points will be divided into *transient* and *recurrent*: and the *natural extension of the ergodic hypothesis* $\Rightarrow$ *all recurrent points are on a single cycle*.

$E(\mathbf{q})$



Evolution drives to unstable manifolds of attracting set.

Points of the discrete attracting set will be imagined located on a selected unstable manifold

After a transient, attractor will appear as union of unstable manifolds.

Probability of each element of $\mathcal{E}$ will simply be the number of points of the attractor in it, since motion is simply 1-cycle perm. of attractor points.

Statistics is *uniquely determined* and just uniform distr. on the attractor points. Unification: Equil&Noneq.

Let σ_E = surface of attractor unstable surfaces (after discretization) inside E and N_E =number of attractor points on it. The condition of invariance of the attractor $\Rightarrow$ consistency on the N_E *i.e.* on the weights w_E.

If "cell" E_i receives points from $E_{i'}$ ($SE_{i'} \cap E_i \neq \emptyset$) and the attractor surface is expanded by a factor $\Lambda(i')$ it must be

$$N_{E_i} = \sum_{i' \to i} \Lambda(i')^{-1} \frac{N_{E'_i}}{\sigma_{E_{i'}}} \sigma_{E_i} \;\Rightarrow\; w_i = \sum_{i' \to i} \Lambda(i')^{-1} w_{i'}$$

An equation that could be formulated without reference to a discretization: $\Rightarrow$ stationary distribution has density on unstable manifolds 1-quely determined by expansion rates, (RPF operator, [7, 8]).

Question: "what happened to the other (uncountably many) invariant distrib. of the continuum map (M, S)?"

Discretization has been done via a regular lattice. Using different discretizations the result would be different and other invariant distributions would arise, in the continuum limit, for the considered system (M, S) and $\neq$ SRB.

Relevance of the SRB distributions is tightly related, [6], to structure of space-time: representability of natural laws via ODE or PDE and via corresponding simulations, is, ultimately, possible because a model of natural laws can be equivalently built on microscpically discrete and regular arrays or on Euclidean continua (a natural law itself).

Such representability makes possible and useful the modern simulations: agreement with observations of simulation results can be seen as a confirmation of a natural law,
As quite explicitly invoked by Boltzmann, *e.g.*[14, p.169]:

Therefore if we wish to get a picture of the continuum in words, we first have to imagine a large, but finite number of particles with certain properties and investigate the behavior of the ensemble of such particles ... one can then assert that they apply to a continuum, and in my opinion this is the only non-contradictory definition of a continuum with certain properties

Ergodicity does not solve the supposed conflict between micr. reversibility and macr. irreversibility.

However Boltzmann (and others, *e.g.*Thomson) had clearly reconciled the two and estimated the minimal time scale on which reversibility could be detected, [15, 16].

The idea of a rather general explanation of the nature of micr. motion, that inspired the founding fathers, is that a general explanation might imply general consequences directly accessible on human time scales, [13].

A micr. property for rather arbitrary systems subject to basic laws of nature (*e.g.*classical mechanics) might be uninteresting in the case of systems with few particles but it might turn out to be a fundamental law for many particle systems.

The example of the second law of thermodynamics, derived by Boltzmann, Clausius, Maxwell, is a paradigmatic example, [3, 4, 5].

A general micr. viewpoint might be very useful to study general properties implied by symmetries:

Thermodynamics reflects the Hamilt. symmetry of the basic equations, both from the viewpoint of periodic recurrence as in [3], via the action principle, or from the ensemble viewpoint in [10], via thermodynamic analogy.

Onsager reciprocity reflects time reversal near equilibrium,

SRB theory is linked to the homogeneity of space time with respect to the Galilei group, as above.

Other micr. symmetries can be reflected macr., once assumed the involved systems to follow a law of motion obeying a general common property.

An example, quite simple and nevertheless non trivial, is the "fluctuation theorem" which reflects the basic time reversal symmetry, [13], of dissipative systems.

A theory of nonequilibrium ensembles can be set up related to time reversal symmetry (V. Lucarini & GG, [17]) Stationary states: $\Rightarrow$ probab. distrib. on phase space $\Rightarrow$ collections of stationary states $\Rightarrow$ ensembles $\mathcal{E}$: in equilibrium give the statistics (canonical, microc., &tc).

Can this be done for stationary nonequilibrium?

Nonequilibrium stationary states are controlled by irreversible micr. equations. But reversibility is fundamental. Hence each evolution should be modelizable by reversible eq.s leading to the same statistics.

The Lorenz96 eq. (periodic b.c.) is a test ground

$$\dot{x}_j = x_{j-1}(x_{j+1} - x_{j-2}) + F - \nu x_j, \qquad j = 0, \ldots, N-1$$

Vary ν: the stationary distrib. μ_ν form an "ensemble" (viscosity ensemble) for L96. $E(x) \stackrel{def}{=} \sum_j x_j^2$ fluctuates in μ_ν. Replace ν by $\alpha(x) = \frac{\sum_i F x_i}{\sum_i x_i^2}$, [17, 18, 19, 20].

New Eq. has $E = E(x) = \sum_i x_i^2$ as exact constant of motion

$$\dot{x}_j = x_{j-1}(x_{j+1} - x_{j-2}) + F - \alpha(x)x_j,$$

Vary E and let μ_E station. distrib.: (energy ensemble). Now volume contracts by $\sigma(x) \sum \partial_j(\alpha(x)x_j)$

$$\sigma(x) = (N-1)\alpha(x), \quad p = \tau^{-1}\int_0^\tau \sigma(x(t))dt/\langle\,\sigma\,\rangle$$

Conjecture,[18, 17]: Equivalent ensembles (under "Chaotic hypothesis"): State $\widetilde{\mu}_E$ labeled by E corresponds to states μ_ν labeled by ν are equivalent if $\mu_E(\alpha(x)) = \nu$ (or x$\mu_\nu(E(x)) = E$). *I.e.* the two ensembles Give the same statistics in the limit of large $R = \frac{F}{\nu^2}$.

Remark the analogy with the equivalence between canonical $(\beta \longleftrightarrow \nu)$ and micro-can. $(E \longleftrightarrow E)$

Why? several reasons '?'. Eg. chaoticity implies

$$\alpha(x(t)) = \frac{\sum_i F x_i}{\sum_i x_i^2} \qquad \text{“self − averaging''}$$

equivalence sets in $(\mathbf{F} \to \infty)$ if $\langle\, \alpha \,\rangle = \nu$ or if $\langle\, E(\cdot) \,\rangle = E$.

Tests performed at $N = 32$ (with checks up to $N = 512$) and high R (at $R > 8$, system is very chaotic with > 20 Lyap.s exponents and at larger R it has $\sim \frac{1}{2}N$ L.e.)

1) $\mu_{\overline{E}}(\alpha) = \nu \longleftrightarrow \mu_\nu(E) = \overline{E}$

2) If g is reasonable (“local”) observable $\frac{1}{T}\int_0^T g(S_t x)dt$ has same statistics in both

3) The “Fluctuation Relation” holds for the fluctuations of phase space vol (reversible case): reflect chaotic hypothesis

4) Found its N-independence and ensemble independence of the Lyapunov spectrum (Livi,Politi,Ruffo)

5) In so doing found or confirmed several scaling and pairing rules for Lyapunov exponents (somewhat surprising) and checked a local version of the F.R.

Scaling of energy-momentum (irreversible model):

$$E = \sum_i x_i^2, \qquad M = \sum_i x_i$$

$$\frac{\overline{E}^i_R}{N} \sim c_E R^{4/3}, \qquad \frac{\overline{M}^i_R}{N} \sim 2c_E R^{1/3} \qquad c_E = 0.59 \pm 0.01$$

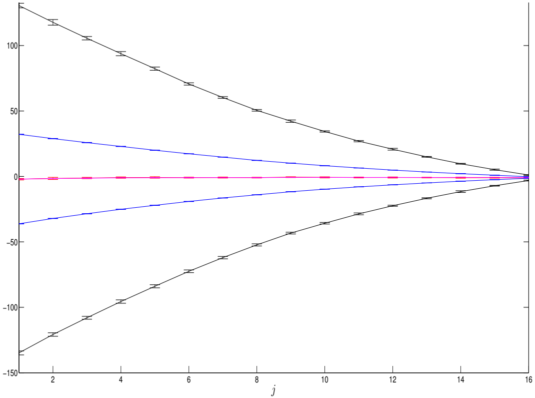
$$\frac{std(E)^i_R}{N} = \frac{\left(\overline{E^2}^i_R - (\overline{E}^i_R)^2\right)^{1/2}}{N} = \tilde{c}_E R^{4/3}, \qquad \tilde{c}_E \sim 0.2 c_E$$

$$\frac{std(M)^i_R}{N} = \tilde{c}_M R^{2/3} \qquad \tilde{c}_E \sim 0.046 \pm 0.001$$

$$t^{i,M}_{dec} \sim c_M R^{-2/3} \qquad c_M = 1.28 \pm 0.01$$

The first two confirm Lorenz96, the 3d,4th "new", 5th is the "decorrelation" time $\langle\, M(t)M(0)\,\rangle$

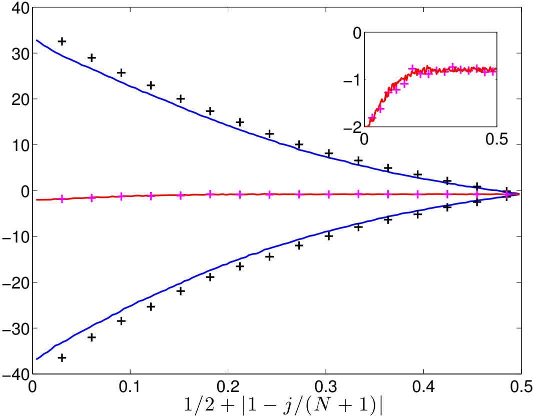
Lyap. exps. arranged pairwise (reversible and irreversible)



$E(\mathbf{q})$

Black: Lyap. exp.s $R = 2048$
Magenta: $\pi(j) = (\lambda_j + \lambda_{N-j+1})/2$.
Blue: Lyap. exp.s $R = 256$
value of $\pi(j)$ at $N = 32$ (invisible below magenta).

Continuous limit of Lyapunov Spectrum (LPR, [21]): asymptotics in $N = 32, 256$ at R fixed:

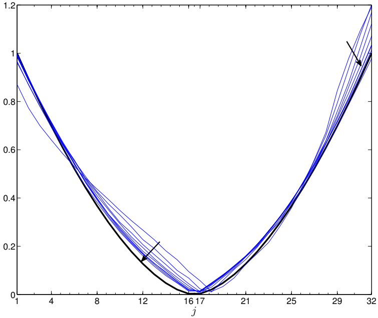


$R = 256$: λ_j for $N = 256$ and Black mark $N = 32$
red line $\pi(j) = (\lambda_j + \lambda_{N-j+1})/2$ for $N = 256$
and marker for $N = 32$; zoom inset

Scaling Lyapunov Spectrum: $8 \le R = 2^n \le 2048$

$$x = \frac{j}{N+1} \;\Rightarrow\; |\lambda(x) + \pi(x)| \sim c_\lambda \, |2x-1|^{5/3} \, R^{2/3}$$

$$\sim |\lambda(x) + 1| \sim c_\lambda \, |2x-1|^{5/3} \, R^{2/3}, \quad c_\lambda \sim 0.8$$



Blue: $|\lambda_j + 1|/(c_\lambda R^{2/3})$, Black: $|2j/(N+1) - 1|^{5/3}$

Dimension of Attractor

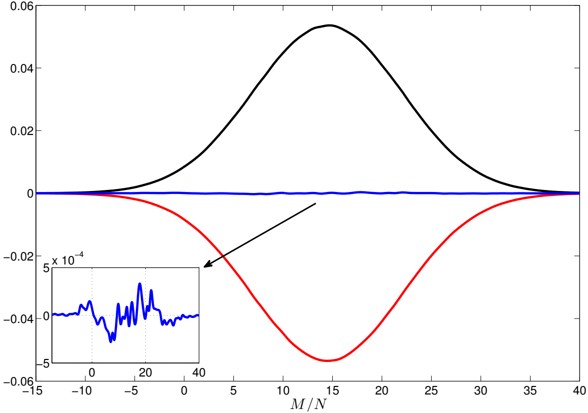
The $|\lambda(x)+1| \sim c_\lambda\, |2x-1|^{5/3}\, R^{2/3}$ yields the full spectrum: hence can compute the KY dimension

$$N - d_{KY} = \frac{N}{1 + c_\lambda R^{\frac{2}{3}}} \xrightarrow[R\to\infty]{} 0, \qquad \forall\ N$$

attractor has a dimension virtually indistinguishable from that of the full phase space.

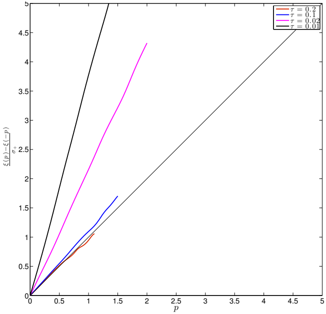
However SRB distribution deeply different from equidistribution (often confused with ergodicity): made clear by the equivalence (if holding) and the validity of the Fluctuation Relation, [22, 23, 1], needs test

Reversible-Irreversible ensembles equivalence:



Black: pdf for M/N rev, $R = 2048$. Blue − pdf for M/N irrev for $R = 2048$. Red black + blue line. Note vertical scales.

Check Fluctuation Relation (FR, [22, 23, 17])



$$p = \frac{1}{\tau} \frac{\int_0^\tau \sigma(x(t))dt}{\langle\sigma\rangle_{srb}}$$

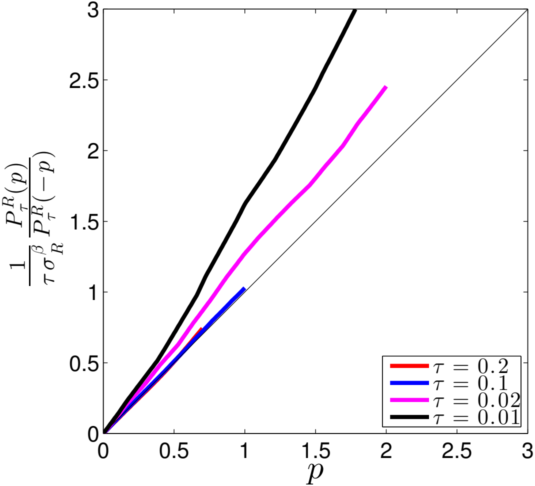
$$\frac{1}{\tau\overline{\sigma}_R} \log \frac{P_\tau^R(p)}{P_\tau^R(-p)} = p \quad ???$$

F.R. slope $c(\tau) \xrightarrow[R\to\infty]{} 1, \; R = 512$

$$c(\tau) = 1 + \left(\frac{t^{r,\sigma}_{dec,R}}{\tau}\right)^{4/3} = 1 + \left(\frac{c_\sigma}{\tau}\right)^{4/3} R^{-8/9}$$

as $\tau \uparrow$ beyond decorrelation time.

Local Fluctuation Relation, [24], $N = 32$



Local F.R.: $R = 2048$. Subsystem of 8 modes, [17]:

$$\frac{1}{\tau} \log \frac{P^R_\tau(p)}{P^R_\tau(-p)} = \overline{\sigma^\beta}_R p + O(\tau^{-1}) = \beta \overline{\sigma}_R p + O(\tau^{-1})$$

References

[1] G. Gallavotti and D. Cohen.

Dynamical ensembles in stationary states.

Journal of Statistical Physics, 80:931–970, 1995.

[2] P. Collet, H. Epstein, and G. Gallavotti.

Perturbations of geodesic flows on surfaces of constant negative curvature and their mixing properties.

Communications in Mathematical Physics, 95:61–112, 1984.

[3] L. Boltzmann.

Über die mechanische Bedeutung des zweiten Hauptsatzes der Wärmetheorie.

Wiener Berichte, 53, (W.A.,#2):195–220, (9–33), 1866.

[4] R. Clausius.

Ueber die Zurückführung des zweites Hauptsatzes der mechanischen Wärmetheorie und allgemeine mechanische Prinzipien.

Annalen der Physik, 142:433–461, 1871.

[5] J. C. Maxwell.

On Boltzmann's theorem on the average distribution of energy in a system of material points.

In: The Scientific Papers of J.C. Maxwell, Cambridge University Press, Ed. W.D. Niven, Vol.2, pages 713–741, 1879.

[6] G. Gallavotti.

Ergodicity: a historical perspective. equilibrium and nonequilibrium.

arxiv: 1604.04239, Submitted:?, 2016.

[7] D. Ruelle.

Chaotic motions and strange attractors.

Accademia Nazionale dei Lincei, Cambridge University Press, Cambridge, 1989.

[8] D. Ruelle.
Turbulence, strange attractors and chaos.
World Scientific, New-York, 1995.

[9] J. P. Eckmann and D. Ruelle.
Ergodic theory of chaos and strange attractors.
Reviews of Modern Physics, 57:617–656, 1985.

[10] L. Boltzmann.
Einige allgemeine sätze über Wärmegleichgewicht.
Wiener Berichte, 63, (W.A.,#19):679–711, (259–287), 1871.

[11] L. Boltzmann.
Studien über das gleichgewicht der lebendigen kraft zwischen bewegten materiellen punkten.

Wiener Berichte, 58, (W.A.,#5):517–560, (49–96), 1868.

[12] G. Gallavotti.

Entropy, nonequilibrium, chaos and infinitesimals: in Bolzmann's legacy, Ed. G.Gallavotti, W.Reiter, Y.Yngvason.

cond-mat/0606477 and Birkhäuser, 2008.

[13] G. Gallavotti.

Nonequilibrium and irreversibility.

Theoretical and Mathematical Physics. Springer-Verlag and http://ipparco.roma1.infn.it & arXiv 1311.6448, Heidelberg, 2014.

[14] L. Boltzmann.

Theoretical Physics and philosophical writings, ed. B. Mc Guinness.

Reidel, Dordrecht, 1974.

[15] L. Boltzmann.

Bemerkungen über einige probleme der mechanischen Wärmetheorie.

Wiener Berichte, 75, (W.A.,#39):62–100, (112–148), 1877.

[16] W. Thomson.

The kinetic theory of dissipation of energy.

Proceedings of the Royal Society of Edinburgh, 8:325–328, 1874.

[17] G. Gallavotti and V. Lucarini.

Equivalence of Non-Equilibrium Ensembles and Representation of Friction in Turbulent Flows: The Lorenz 96 Model.

Journal of Statistical Physics, **156**, *1027-1065, 2014*, 156:1027–10653, 2014.

[18] G. Gallavotti.

Extension of Onsager's reciprocity to large fields and the chaotic hypothesis.

Physical Review Letters, 77:4334–4337, 1996.

[19] G. Gallavotti.

Equivalence of dynamical ensembles and Navier Stokes equations.

Physics Letters A, 223:91–95, 1996.

[20] G. Gallavotti.

New methods in nonequilibrium gases and fluids.

Open Systems and Information Dynamics, 6:101–136, 1999.

[21] R. Livi, A. Politi, and S. Ruffo.

Distribution of characteristic exponents in the thermodynamic limit.

Journal of Physics A, 19:2033–2040, 1986.

[22] G. Gallavotti and D. Cohen.

Dynamical ensembles in nonequilibrium statistical mechanics.

Physical Review Letters, 74:2694–2697, 1995.

[23] G. Gallavotti.

Reversible Anosov diffeomorphisms and large deviations.

Mathematical Physics Electronic Journal (MPEJ), 1:1–12, 1995.

[24] G. Gallavotti.

A local fluctuation theorem.

Physica A, 263:39–50, 1999.