Renormalization group, Kondo effect and hierarchical models

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1-d lattice, fermions+impurity, "Kondo problem"

$$H_h = \sum_{x=-L/2}^{L/2-1} \psi^+(x)\,(-\frac{1}{2}\Delta - 1)\,\psi^-(x) + h\ \tau^z$$

$$H_K = H_h + \lambda\psi^+(0)\sigma^j\psi^-(0)\,\tau^j \ = \ H_h + V$$

(1) $\psi^\pm_\alpha(x)$ C&A operators, σ^j, τ^j, $j = 1, 2, 3$, Pauli matrices

(2) $x \in$ unit lattice, $-L/2$, $L/2$ identified (periodic b.c.)

(3) $\Delta f(x) = f(x+1) - 2f(x) + f(x-1)$ discrete Laplacian.

No interaction $(\lambda = 0)$: 1 impurity and $\beta h < 1$ (*e.g.* $h = 0$)

$$\chi(\beta, h) \propto \beta \xrightarrow[\beta\to\infty]{} \infty, \qquad \forall\, L \geq 1,\ \beta h < 1$$

Interaction (classical) 1 elec.&1 impurity:
1) field on impurity & $\lambda \neq 0$

$$\chi(\beta, 0) = 0 \quad \text{repulsive}, \qquad +\infty \quad \text{attractive}$$

2) Still true if $L < \infty$ classic&quantum or $L = \infty$ classic

BUT

If $L = \infty$ quantum chain: new phenomena

1) no impurity: $\Rightarrow$ Pauli paramagnetism (1926)

local (or specific) magnetic suscept. $< \infty$ at $T \geq 0$:

$$\chi(\infty, 0) = \rho \frac{1}{k_B T_F} \frac{d}{2}, \qquad (Pauli)$$

2) at fixed $\lambda < 0 \Rightarrow$ Kondo effect:

susceptibility $\chi(\beta, h)$

smooth and > 0 at $T = 0$ and $h \geq 0$

Kondo realized the problem (3^d-order P.T.) and gave arguments (1964) for $\chi < \infty$ (actually conductivity $< \infty$)

Anderson-Yuval-Hamann (1969,70) $\Rightarrow$ multiscale nature, relation with 1D Coulomb gas & (no Kondo eff. $\lambda > 0$), &

& stress lack of asymptotic freedom = obstacle for $\lambda < 0$.

Wilson (1974-1975) had overcome lack of asympt. freedom: simplified model and a recursion scheme, $\frac{1}{2}$-numerically.

Andrei (1980): exact solution of closely related model.

Method builds sequence of approximate Hamiltonians more and more accurately representing the system on larger and larger scales, with Kondo effect via a nontrivial fixed point.

Evaluate $Z = \mathrm{Tr}\, e^{-\beta H_K}$ via Wick's rule.

$$Z = \mathrm{Tr} \left\langle \sum_{n=0}^{\infty} (-1)^n \int_{0<t_1<\cdots<t_n<\beta} dt_1 \cdots dt_n V(t_1) \cdots V(t_n) \right\rangle$$

$$V(t) \stackrel{def}{=} -\lambda_0 \psi^+(t) \sigma^j \psi^-_{\alpha_2}(t)\, \tau^j - h\, \boldsymbol{\omega}_j \tau^j$$

Averages of observables depending only on the site 0 (*e.g.* impurity susceptibility) require by Wick $\Rightarrow$ only Feynman graphs with propagators at $x = 0$: $g(t - t')$:

$$g(t - t') = \sum_{\omega=\pm} \int \frac{dk_0 dk}{(2\pi)^2} \frac{e^{ik_0(t-t')}}{-ik_0 + \omega k} \chi(k_0^2 + k^2),$$

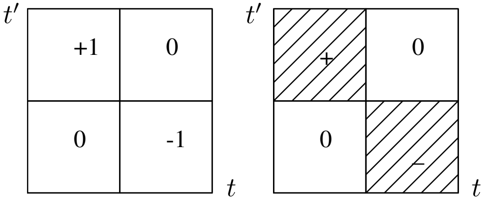
here a first simplification: cut-off of the large k, k_0 and linear dispersion relation $\pm k$ at the Fermi level $k = 0$).
The multiscale decomposition of g

$$g(t - t') = \sum_{m=0}^{-\infty} 2^m g_0(2^m(t - t'))$$

exhibits the scaling properties of g: namely the long range $\sim \frac{1}{t-t'}$ decomposed as a sum of short range propagators identical up to scaling.

The hierarchical model introduces a further simplification

$$
\begin{aligned}
g(t-t') &= \sum_{m=0}^{-\infty} 2^m g_0(2^m(t-t')) \\
g_0(t,t') &= 0 \text{ unless } t,t' \in [n2^{-m},(n+1)2^{-m}] \\
g_0(t,t') &= \begin{cases} 1 & \text{if } t \in [n2^{-m},(n+\frac{1}{2})2^{-m}] \text{ and } t' > (n+\frac{1}{2})2^{-m} \\ -1 & \text{if } t' \in [n2^{-m},(n+\frac{1}{2})2^{-m}] \text{ and } t > (n+\frac{1}{2})2^{-m} \end{cases} \\
g_0(t,t') &= 0 \qquad \text{otherwise}
\end{aligned}
$$



g_0 loses translation invariance but the propagator g keeps the multiscale and long range properties of the initial model, at least hierarchically

But since the impurity is localized observ. localized at 0 depend on fields at 0, $\psi^{\pm}(0), \varphi^{\pm} \Rightarrow$ **1D problem** (AYH).

Illustration of (AYH970) remark: 1D problem, (long range)
Main operators in the Lagrangian:

$$O_0(t) \stackrel{def}{=} \psi^+(t)\boldsymbol{\sigma}\psi^-(t)\cdot\boldsymbol{\tau} = \vec{A}(t)\cdot\boldsymbol{\tau}, \qquad O_5(t) \stackrel{def}{=} \boldsymbol{\tau}\cdot\boldsymbol{\omega}$$

(in Grassmannian form) and
$\mathcal{L}_K$ on scale m is (with $\alpha_0 < 0, \alpha_5 = h \geq 0$ else 0).

$$\int e^{\mathcal{L}_K^{[<=m]}(\psi^{[\leq m]}} d\psi = \int e^{-\int_0^\beta \sum_i \alpha_i^{[m]} O_i(t)\, dt} d\psi^{[0]} d\psi^{[1]} \dots d\psi^{[m+1]}$$

Set RG analysis via (Grassmannian) for $\mathrm{Tr}\, e^{-\beta H_K}$

Key: IF $h = 0$ then $\mathcal{L}_K^{[m]}(t)$ is $\forall m$:

$$\alpha_0^{[m]} O_0(t)\cdot\boldsymbol{\tau} + \alpha_1^{[m]} O_1(t)$$

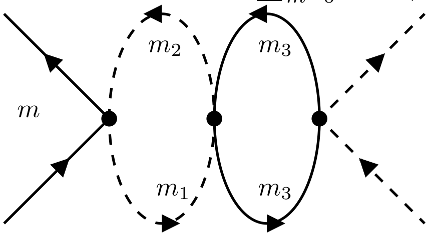
i.e. no new operators needed at any scale (exact recursion)

Scaling O_0 = marginal, O_1 irrrelevant,O_5 = relevant

The RG consists in

1) Expand perturbatively $Z^{[>m]} = e^{V^{[m]}}$ via Feynman gr.
heavily using the hierarchical structure

2) Decompose propagators as $\sum_{m=0}^{-\infty} 2^m g_0(2^m(t-t'))$



3) Recognize: at $h = 0$ no new operators can arise besides

$$O_4 = \vec{A}\cdot\vec{h},\ O_5 = \boldsymbol{\sigma}\cdot\vec{h},\ O_6 = \vec{A}\cdot\vec{h}\,\boldsymbol{\sigma}\cdot\vec{h}, O_7 = \vec{A}^2\boldsymbol{\sigma}\cdot\vec{h},$$

3) Recognize that the result contains a few series that can collected to form a sequence of running couplings

$$\boldsymbol{\alpha}^{[m]} = (\alpha_0^{[m]}, \alpha_1^{[m]}, \alpha_4^{[m]}, \alpha_5^{[m]}, \alpha_6^{[m]}, \alpha_7^{[m]}).$$

with only $\alpha_0^{[m]}, \alpha_1^{[m]} \neq 0$ if $h = 0$

4) Each is a convergent series in the initial couplings α_0, h, if small enough (BUT converg. radius m dependent)

5) Recognize that the $\boldsymbol{\alpha}^{[m]}$ satisfy a formal recursion

$$\boldsymbol{\alpha}^{[m]} = \Lambda \boldsymbol{\alpha}^{[m+1]} + \mathcal{B}(\boldsymbol{\alpha}^{[m+1]})$$

and $\mathcal{B}$ can be expressed as a "polynomial" with coefficients which are geometric series in $\boldsymbol{\alpha}^{[m+1]}$; $\Lambda = (1, \frac{1}{2}, 1, 2, 1, \frac{1}{2})$.

Even forgetting convergence, PT of no use: marginal term grows (if $\lambda_0 < 0$) and generates growing ("relevant" terms)!

6) Sum the geometric series to obtain a closed from of $\mathcal{B}$. After a natural change of variables $\boldsymbol{\alpha} \longleftrightarrow \boldsymbol{\lambda}$ at $h = 0$

$$\begin{aligned}
\lambda_0' =& \frac{1}{C}(\lambda_0 + 3\lambda_0\lambda_1 - \lambda_0^2) \\
\lambda_1' =& \frac{1}{C}(\frac{1}{2}\lambda_1 + \frac{1}{8}\lambda_0^2), \\
C =& 1 + \frac{3}{2}\lambda_0^2 + 9\lambda_1^2
\end{aligned}$$

Non perturbative: for $m \to -\infty$ (IR limit, $\beta = +\infty$, $T = 0$)

$\boldsymbol{\lambda}^{[m]}, \boldsymbol{\alpha}^{[m]}$ converge to non trivial fixed point

if $h = 0, \alpha_0 < 0$, exactly computable,
$\lambda_0^* = -7.807257...10^{-1}, \lambda_1^* = 5.292875...10^{-2}$

$$\lambda_0^* = -x\frac{1+5x}{1-4x}, \;\; \lambda_1^* = \frac{x}{3}, \;\; x = 7.807257...10^{-1},$$

with $4 - 19x - 22x^2 - 107x^3 = 0$, real root.

Susceptibility: new operators needed to close beta

$$O_4 = \vec{A} \cdot \vec{h},\ O_5 = \boldsymbol{\sigma} \cdot \vec{h},\ O_6 = \vec{A} \cdot \vec{h}\, \boldsymbol{\sigma} \cdot \vec{h}, O_7 = \vec{A}^2 \boldsymbol{\sigma} \cdot \vec{h},$$

O_0, O_4, O_6 marginal, O_5 relevant, O_1, O_7 irrelevant

Calculating beta function: via Feynman graphs, after simplifications, a beta function with 36 coeff is found

From the flow of the $\boldsymbol{\alpha}$ the partition function $Z(\beta, h)$ is computed and susceptibility

$$\chi(\beta, h) = \partial_h^2 \log Z(\beta, h)$$

follows as a function of h.

The beta function is a rational function defined by the ratio of two polynomials of degree 2.

$$
\begin{aligned}
C =&1 + \lambda_0^2 + \frac{1}{2}(\lambda_0 + \lambda_6)^2 + 9\lambda_1^2 + \frac{1}{2}\lambda_4^2 + \frac{1}{4}\lambda_5^2 + 9\lambda_7^2 \\
\lambda_0' =&\frac{1}{C}(\lambda_0 - \lambda_0^2 + 3\lambda_0\lambda_1 - \lambda_0\lambda_6) \\
\lambda_1' =&\frac{1}{C}(\frac{1}{2}\lambda_1 + \frac{1}{8}\lambda_0^2 + \frac{1}{12}\lambda_0\lambda_6 + \frac{1}{24}\lambda_4^2 + \frac{1}{4}\lambda_5\lambda_7 + \frac{1}{24}\lambda_6^2) \\
\lambda_4' =&\frac{1}{C}(\lambda_4 + \frac{1}{2}\lambda_0\lambda_5 + 3\lambda_0\lambda_7 + 3\lambda_1\lambda_4 + \frac{1}{2}\lambda_5\lambda_6 + 3\lambda_6\lambda_7) \\
\lambda_5' =&\frac{1}{C}(2\lambda_5 + 2\lambda_0\lambda_4 + 36\lambda_1\lambda_7 + 2\lambda_4\lambda_6) \\
\lambda_6' =&\frac{1}{C}(\lambda_6 + \lambda_0\lambda_6 + 3\lambda_1\lambda_6 + \frac{1}{2}\lambda_4\lambda_5 + 3\lambda_4\lambda_7) \\
\lambda_7' =&\frac{1}{C}(\frac{1}{2}\lambda_7 + \frac{1}{12}\lambda_0\lambda_4 + \frac{1}{4}\lambda_1\lambda_5 + \frac{1}{12}\lambda_4\lambda_6)
\end{aligned}
$$

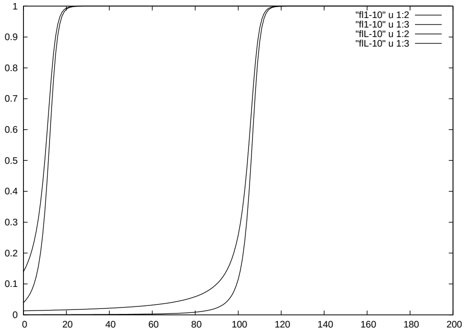


Fig.2: plot of $\frac{\lambda_i}{\lambda_i^*}$, $i = 0, 1$, as a function of $N_\beta = \log_2 \beta$, $\lambda_0 \equiv \alpha_0 = -0.1, -0.01$ respectively the left and the right pairs.

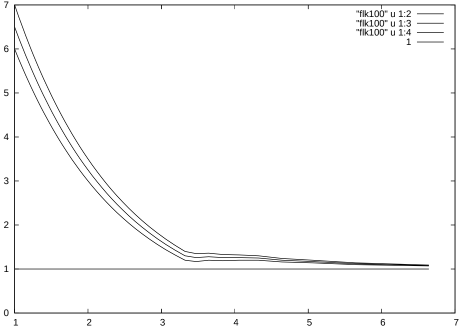


Fig.3: inflection point $n_0(\lambda_0)$: $n_0(\lambda_0) \cdot |\lambda_0|$ vs. $|\log_2|\lambda_0||$: only data with 10% error (upper and lower curves) visual lines interpolate data

$$T_K = \, const\, e^{-c_0 \lambda_0^{-1}}$$

For $h \neq 0$ the flow leads to "high T fixed pt." at scale $\propto 1/|\log h|$

The equation of state

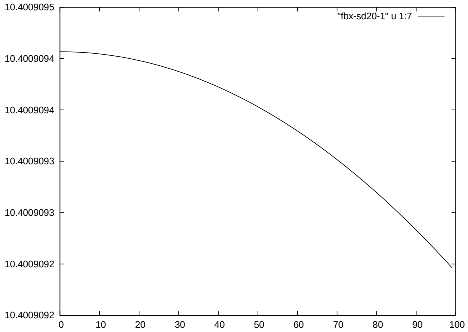


Fig.4: plot of $\chi(\beta, h)$ for $h \in [0, 10^{-6}]$ at $\lambda_0 = -0.3$ and $\beta = 2^{20}$ (so that the largest value for βh is ~ 1)

[2, 3, 5, 4, 6]

It is interesting to compare the above results with the ones that would be given by 2^d or 3^d perturbation theory, in $h = 0$ field (for simplicity).

Just expand the exact beta function in powers

(1) to order 2 the flow diverges (strong coupling)
(2) to order 3 (and very likely to all orders) the flow converges to a non trivial fixed point
(3) the magnetic susceptibility diverges: *i.e.* a nontrivial fixed point does not necessarily imply a Kondo effect

This exhibits the key difficulty that is met in treating the $s-d$-model (non hierarchical) via the RG.

On the one hand it should be possible to apply the local (*i.e.* "Roman") methods to prove that the beta function is well defined and convergent for small running couplings.

On the other hand the radius of convergence would be necessarily small (as it depends on the bounds of [7, 8]): and to third order is likely to yield the hierarchical result with the fixed point existing but located out of the convergence radius.

The theory of a version of the Anderson model without approximations rests on the exact result of [1]; and a rigorous version of the RG analysis discovered by Wilson, [9] is certainly an interesting open problem.

An interesting graphical representation of the RG flow in the hierarchical model has been developed by J. Jauslin.

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