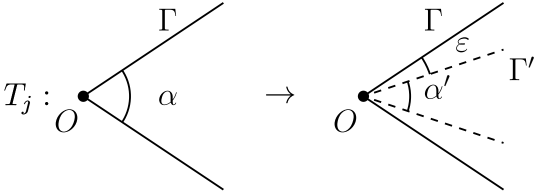
Random matrices and rarefied gases properties

Cone property: enjoyed by $\ldots, T_{-1}, T_0, T_1, T_2, \ldots$ of $d \times d$ real invertible matrices if



i.e. $T_j\Gamma \subset \Gamma'$ and $\Gamma'/O \subset$ of interior Γ^0 of Γ

Let $H_{n,N} v_N \overset{def}{=} T_n T_{n+1} \cdots T_N \, v_N,\ v_N \in \Gamma$ asymptotically prop. to λ^{N-n} with λ a "Lyapunov coefficient"?.

λ is an "eigenvalue". Maybe exists covariant $h(n)$? and $\Lambda(n,p)$ analytic?, s.t. ([6])

$$h(n) \,=\, \Lambda(n,p)\, T_n \ldots T_{p-1} h(p), \qquad n < p$$

Theorem 1: (1) Let $||T_n|| < B_0$, $B_0 > 0$. Then given any sequence $\{v_j\}_{j\geq 1}, ||v_j|| = 1, v_j \in \Gamma$, $\exists \overline{\Lambda}_{v_N}(n, N)$, et $\forall \{v_j\}_{j\geq 1}$

$$h(n) = \lim_{N\to\infty} \frac{T_n T_{n+1} \cdots T_N \, v_N}{\overline{\Lambda}_{v_N}(n, N)}, \qquad \forall 1 \leq n \leq p$$

$$\Lambda(n, p) = \lim_{N\to\infty} \frac{\overline{\Lambda}_{v_N}(p, N)}{\overline{\Lambda}_{v_N}(n, N)} > 0, \qquad \forall 1 \leq n \leq p$$

(2) Λ, h is analytic for small variations of entries.

(3) The $h(n)$ are "eigenvectors" *i.e.* $\Lambda(n, p)$ **s.t.:**

$$T_{p-1}^{-1} \ldots T_n^{-1} \, h(n) \;=\; \Lambda(n, p) \, h(p)$$

(4) $\exists B$ s.t. $B^{-1} \leq ||h(n)|| \leq B$ and $\limsup, \liminf$ of $p^{-1} \log|\Lambda(n, p)|$, as $p \to \infty$, are n independent.

$$
\begin{aligned}
&(T_n T_{n+1} \cdots T_N v_N)_{i_n} \\
&= \sum_{i_{n+1},\ldots,i_N} ((T_n)_{i_n,i_{n+1}} (T_{n+1})_{i_{n+1},i_{n+2}} \ldots (T_N)_{i_N,i_{N+1}} v_{i_{N+1}})
\end{aligned}
$$

looks "partition function with boundary conditions" i_n, v.

If $(T_k)_{i,j} > 0$ then $(T_k)_{i,j} = e^{-V_k(i,j)}$ with V_k "random".

Interpretation of "spin glass"; but $(T_k)_{i,j}$ may be $\neq > 0$

Pursue the idea: diagonalize $T_n = \sum_{\sigma=0}^{d-1} \lambda_{n,\sigma} \, |n,\sigma\rangle\langle n,\sigma|$

$$
\begin{aligned}
&H_{n,N} v_N \stackrel{def}{=} T_n T_{n+1} \cdots T_N \, v_N \\
&= \sum_{\sigma_n,\ldots,\sigma_N} |n,\sigma_n\rangle \cdot \Big(\prod_{j=n}^{N} \lambda_{j,\sigma_j} \Big) \prod_{j=n}^{N} \langle j,\sigma_j | j+1, \sigma_{j+1}\rangle
\end{aligned}
$$

Here $|N+1, \sigma_{N+1}\rangle \equiv v$ (σ_{N+1} is label, not sum index)

Call $(\sigma_n, \ldots, \sigma_N)$ a "spin config". **Isolate** the $\sigma_j = 0$.

$$0\ \ 0\ \ \underline{1\ \ 1}\ \ 0\ \ 0\ \ 0\ \ \underline{1\ \ 2\ \ 1}\ \ 0$$

Fig.2

and form intervals $J = [h, h'] = (h, h+1, \ldots, h')$, $n \le h \le h' \le N$ and, on each, *spin configurations* $\boldsymbol{\sigma}_J$.

$$H(n, N)_{\sigma_{N+1}=0}$$

$$= const \sum_{\sigma_n, \ldots, \sigma_N} |n, \sigma_n\rangle \cdot \Big(\prod_{j=n}^{N} \frac{\lambda_{j,\sigma_j}}{\lambda_{j,0}} \Big) \prod_{j=n}^{N} \frac{\langle j, \sigma_j | j+1, \sigma_{j+1}\rangle}{\langle j, 0 | j+1, 0\rangle}$$

This is (simplifying contributions of stretches with $\sigma = 0$)

$$H_{n,N} v_N = \overline{\Lambda}(n, N) \sum_{\substack{J_1 < \ldots < J_s \\ s \ge 0}} \sum_{\boldsymbol{\sigma}_{J_1}, \ldots, \boldsymbol{\sigma}_{J_s}} |n, \sigma_n\rangle \frac{\prod_{i=1}^{s} I(J_i, \boldsymbol{\sigma}_{J_i})}{\Omega(n, N)}$$

$$H(n,N)_{\sigma_{N+1}=0}$$

$$= \prod \lambda_{j,0} \sum_{\sigma_n,\ldots,\sigma_N} |n,\sigma_n\rangle \cdot \Big(\prod_{j=n}^{N} \frac{\lambda_{j,\sigma_j}}{\lambda_{j,0}}\Big) \prod_{j=n}^{N} \langle j,\sigma_j | j+1, \sigma_{j+1}\rangle$$

$$H(n,N)_{\sigma_{N+1}=0}$$

$$= (\prod \lambda_{j,0} \langle j,0|j+1,0\rangle)\Omega(n,N)$$

$$\cdot \sum_{\sigma_n,\ldots,\sigma_N} \frac{|n,\sigma_n\rangle}{\Omega(n,N)} \Big(\prod_{j=n}^{N} \frac{\lambda_{j,\sigma_j}}{\lambda_{j,0}}\Big) \prod_{j=n}^{N} \frac{\langle j,\sigma_j|j+1,\sigma_{j+1}\rangle}{\langle j,0|j+1,0\rangle}$$

$$\Omega(n,N) \stackrel{def}{=} \sum_{s\geq 0} \sum_{\substack{J_1<\ldots<J_s \\ \boldsymbol{\sigma}_{J_1},\ldots,\boldsymbol{\sigma}_{J_s}}} \prod_{i=1}^{s} I(J_i, \boldsymbol{\sigma}_{J_i})$$

$(H_{n,N} v_N)_\sigma =$ is therefore $\overline{\Lambda}(n,N) \sum_{\sigma_n=\sigma} P_{n,\sigma,N}\, |n,\sigma\rangle$

where if $J = [h, h']$:

$$I(J, \boldsymbol{\sigma}_J) \stackrel{def}{=} \Big(\prod_{j=h}^{h'} \frac{\lambda_{j,\sigma_j}}{\lambda_{j,0}} \Big) \Big(\prod_{j=h}^{h'-1} \frac{\langle j, \sigma_j | j+1, \sigma_{j+1} \rangle}{\langle j, 0 | j+1, 0 \rangle} \Big)$$

$$\cdot \Big(\frac{\langle h-1, 0 | h, \sigma_h \rangle}{\langle h-1, 0 | h, 0 \rangle} \Big)^{\delta_{h>n}} \Big(\frac{\langle h', \sigma_{h'} | h'+1, 0 \rangle}{\langle h', 0 | h'+1, 0 \rangle} \Big)$$

$$\overline{\Lambda}(n, N) \stackrel{def}{=} \Omega(n, N) \prod_{j=n}^{N} \Big(\lambda_{j,0} \, \langle j, 0 | j+1, 0 \rangle \Big)$$

$$\Omega(n, N) \stackrel{def}{=} \sum_{s \geq 0} \sum_{\substack{J_1 < \ldots < J_s \\ \boldsymbol{\sigma}_{J_1}, \ldots, \boldsymbol{\sigma}_{J_s}}} \prod_{i=1}^{s} I(J_i, \boldsymbol{\sigma}_{J_i})$$

The $(H_{n,N} v_N)_\sigma$ is therefore $P_{n,\sigma,N} |n, \sigma\rangle \; P_{n,\sigma,N}$ equal to the "probability" that at site n the "spin" is σ in the model with polymers activity $I(J, \boldsymbol{\sigma}_J)$

Bounds (*gap* $\frac{|\lambda_j|}{|\lambda_0|} \le \gamma$, cone property "*inclination* α', *polymers number* $(d-1)$ per base site,

$$\prod \frac{|\lambda_j|}{|\lambda_0|} \le \gamma^{h'-h+1}, \prod \frac{\langle j, \sigma_j | j+1, \sigma_{j+1}\rangle}{\langle j, 0 | j+1, 0\rangle} \le \frac{1}{(\cos\alpha')^{h'-h+1}}$$

$$\sum_{\boldsymbol{\sigma}_J} |I(J, \boldsymbol{\sigma}_J)| \le (\frac{(d-1)\gamma}{\delta})^{h'-h+1} \equiv \eta^{h'-h+1}, \qquad \delta \stackrel{def}{=} \cos\alpha'$$

The $(H(n,N)v)$ can be written: $\sum_{\sigma=0}^{d-1} P_{n,\sigma,N} |n,\sigma\rangle$ with

$$P_{n,\sigma,N} \stackrel{def}{=} \frac{\Omega_\sigma(n,N)}{\Omega(n,N)} \stackrel{def}{=} \sum_{s\ge 0} \sum^{*\sigma} \frac{\prod_{i=1}^{s} I(J_i, \boldsymbol{\sigma}_{J_i})}{\Omega(n,N)}$$

the $*\sigma$ indicates sum restricted to polymers with $\sigma_n = \sigma$.

Polymers gas on $[n, N]$ with activity $|I(S, \boldsymbol{\sigma}_S)| < \eta^{|J|}$

Actually "polymers" $\equiv$ "segments J decorated $\boldsymbol{\sigma}_J$"

Fisher droplet model (machine for (counter)examples)

Cluster expansion converges if:

$$\mu \stackrel{def}{=} \sup_{J,\boldsymbol{\sigma}_J} |I(J, \boldsymbol{\sigma}_J)|^{\frac{1}{2}} \exp \sum_{S,\boldsymbol{\sigma}_S}^{*} |I(S, \boldsymbol{\sigma}_S)|^{\frac{1}{2}} < 1$$

sum $* \Rightarrow S \cap J \neq \emptyset$. *i.e. if* $\eta = (d-1)\delta^{-1}\gamma$ *small enough*

$$P_{n,\sigma,N} = \sum_{J \ni n, \boldsymbol{\sigma}_J, \sigma_n = \sigma} \varphi^T(J, \boldsymbol{\sigma}_J), \quad \sum_{\boldsymbol{\sigma}_J} |\varphi^T(J, \boldsymbol{\sigma}_J)| < C\eta^{\frac{1}{2}|J|}$$

with the sum exponentially convergent to a boundary condition independent limit.

Necessary γ=gap small enough: if not replace T_k by $T_k T_{k+1} \cdots T_{k+q}$ with q large enough! ...

Application to Dynamical systems

$\mathcal{F}$ smooth bounded manif., $\tau : \mathcal{F} \to \mathcal{F}$ smooth, smooth inverse, map on $\mathcal{F}$. Let $T(x)E(x) = E(\tau x)$.

At each point $x \in \mathcal{F}$ closed convex cones $\Gamma(x) \supset \Gamma'(x)$, apex at x in a space $E(x)$ (dimens. d).

Min. angle $\Gamma(x) - \Gamma'(x)$ = "inclination" $\varepsilon(x)$
Max. angles in $\Gamma(x), \Gamma'(x)$ = "openings" be $\theta(x)$,$\theta'(x)$.

Definition: *Let $T(x)$ depend on a variable z, real analytic for $|z| < \nu$; $T(x)\Gamma(x) \subset \Gamma'(\tau x)$ for z real, $|z| < \nu$, then T "ν'-z-analytic with cone p.", $\nu' \le \nu$.* **Let** $T_n \stackrel{def}{=} T(\tau^{-n} x)$**.**

Theorem 2: $x \to v(x) \in \Gamma'(x), ||v(x)|| \equiv 1$ **measurable**
(1) $\exists$ continuous functions $x \to b(x) \in \Gamma(x)$, and $x \to \Lambda(x,p), p = 0, 1, \ldots$, s.t. for all $p > 0, x \in \mathcal{F}, v$, $\exists$

$$b(x) = \lim_{N\to\infty} \frac{T(x)\cdots T(\tau^{-(N-1)}x)|v(\tau^{-N}x)\rangle}{\overline{\Lambda}(x,N)}$$

$$\Lambda(x,p) = \lim_{N\to\infty} \frac{\overline{\Lambda}(\tau^p x, N)}{\overline{\Lambda}(x,N)} > 0, \qquad \forall v$$

(2) $b(x)$ **are "covariant" or "eigenvectors" for products of** $T(\tau^{-j}x)$ **in the sense**

$$T^{-1}(\tau^{-(p-1)}x)\ldots T^{-1}(x)b(x) \;=\; \Lambda(x,p)\, b(\tau^{-p}x)$$

(3) $b(x), \Lambda(x,p)$ **are** v**-indep., continuous and** $\exists B$ **s.t.** $B^{-1} < ||b(x)|| < B$**; the** $\frac{b(x)}{||b(x)||}$ **def. the "+ direction", at** τx**.**

(4) $\Lambda(x,p)$ *analytic in* $z, |z| < \nu$*.*

(5) $\limsup_{p\to\infty} \frac{1}{p} \log|\Lambda(x,p)|$ *and* $\liminf$ *are constants of motion (k-independent at* $\tau^k x$*).*

Theorem 3: *Let ρ be τ–invariant distribution on $\mathcal{F}$ Then:*
(1) the limits are ρ-a.e. equal (FK nor Ra not needed);
(2) ρ ergodic $\Rightarrow$ also ρ–a.e. x-independent.

Consequence

If ρ is ergodic the logarithm $= -\lambda_+$, analytic in z.

Proof essence: understand case T_j diagonalizable, real eigenvectors and eigenvalues uniformly (in j) pairwise separated; furthermore

$$\max_{n,\sigma=1,\ldots,d-1} |\lambda_{\sigma,n}|/|\lambda_{0,n}| < \varepsilon_0, \qquad \varepsilon_0 \text{ small}$$

General case via $T'_n = T_{nq}T_{nq+1}\cdots T((n+1)q-1)$: then q large enough $\to$ large gap (& no diagonalization need). Theorem holds for T'_n hence for T_n.

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