

Renormalization group, Kondo effect and hierarchical models

G.Benfatto, I.Jauslin & GG

1-d lattice, fermions+impurity, “Kondo problem”

$$H_h = \sum_{x=-L/2}^{L/2-1} \psi^+(x) \left(-\frac{1}{2}\Delta - 1\right) \psi^-(x) + h \tau^z$$
$$H_K = H_0 + \lambda \psi^+(0) \sigma^j \psi^-(0) \tau^j = H_h + V$$

- (1) $\psi_\alpha^\pm(x)$ C&A operators, $\sigma^j, \tau^j, j = 1, 2, 3$, Pauli matrices
- (2) $x \in$ unit lattice, $-L/2, L/2$ identified (periodic b.c.)
- (3) $\Delta f(x) = f(x+1) - 2f(x) + f(x-1)$ discrete Laplacian.

No interaction ($\lambda = 0$): 1 impurity and $\beta h < 1$ (e.g. $h = 0$)

$$\chi(\beta, h) \propto \beta \xrightarrow{\beta \rightarrow \infty} \infty, \quad \forall L \geq 1, \beta h < 1$$

Interaction (classical) 1 elec.&1 impurity:

1) field on impurity & $\lambda \neq 0$

$$\chi(\beta, 0) = 0 \quad \text{repulsive}, \quad +\infty \quad \text{attractive}$$

2) Still true if $L < \infty$ classic&quantum or $L = \infty$ classic

BUT

If $L = \infty$ quantum chain: new phenomena

1) no impurity: $\Rightarrow$ Pauli paramagnetism (1926)

local (or specific) magnetic suscept. $< \infty$ at $T \geq 0$:

$$\chi(\infty, 0) = \rho \frac{1}{k_B T_F} \frac{d}{2}, \quad (Pauli)$$

2) at fixed $\lambda < 0 \Rightarrow$ Kondo effect:

susceptibility $\chi(\beta, h)$

smooth and > 0 at $T = 0$ and $h \geq 0$

Kondo realized the problem (3^d -order P.T.) and gave arguments (1964) for $\chi < \infty$ (actually conductivity $< \infty$)

Anderson-Yuval-Hamann (1969,70) $\Rightarrow$ multiscale nature, relation with 1D Coulomb gas & (no Kondo eff. $\lambda > 0$), &

& stress **lack of asymptotic freedom** = obstacle for $\lambda < 0$.

Wilson (1974-1975) had overcome lack of asympt. freedom: simplified model and a **recursion scheme**, $\frac{1}{2}$ -numerically.

Andrei (1980): exact solution of closely related model.

Method builds **sequence of approximate** Hamiltonians more and more accurately representing the system on larger and larger scales, with Kondo effect via a **nontrivial fixed point**.

Evaluate $Z = \text{Tr } e^{-\beta H_K}$ via Wick's rule.

$$Z = \text{Tr} \left\langle \sum_{n=0}^{\infty} (-1)^n \int_{0 < t_1 < \dots < t_n < \beta} dt_1 \dots dt_n V(t_1) \dots V(t_n) \right\rangle$$
$$V(t) \stackrel{\text{def}}{=} -\lambda_0 \psi^+(t) \sigma^j \psi_{\alpha_2}^-(t) \tau^j - h \omega_j \tau^j$$

Averages of observables depending only on the site 0 (*e.g.* impurity susceptibility) require by Wick $\Rightarrow$ **only Feynman graphs with propagators at $x = 0$** : $g(t - t')$:

$$g(t - t') = \sum_{\omega=\pm} \int \frac{dk_0 dk}{(2\pi)^2} \frac{e^{ik_0(t-t')}}{-ik_0 + \omega k} \chi(k_0^2 + k^2),$$

here a first simplification: **cut-off of the large k, k_0 and linear dispersion relation $\pm k$** at the Fermi level $k = 0$).
The multiscale decomposition of g

$$g(t - t') = \sum_{m=0}^{-\infty} 2^m g_0(2^m(t - t'))$$

exhibits the scaling properties of g : namely the **long range** $\sim \frac{1}{t-t'}$ decomposed as a sum of short range propagators **identical up to scaling**.

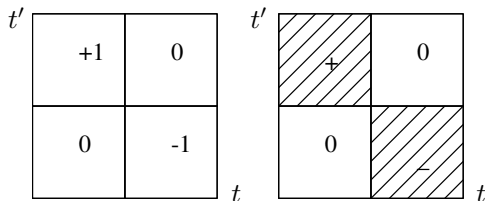
The hierarchical model introduces a **further simplification**

$$g(t - t') = \sum_{m=0}^{-\infty} 2^m g_0(2^m(t - t'))$$

$$g_0(t, t') = 0 \text{ unless } t, t' \in [k, k + 1]$$

$$g_0(t, t') = \begin{cases} 1 & \text{if } t \in [k, k + \frac{1}{2}] \text{ and } t' \in [k + \frac{1}{2}, k] \\ -1 & \text{if } t' \in [k, k + \frac{1}{2}] \text{ and } t \in [k + \frac{1}{2}, k] \end{cases}$$

$$g_0(t, t') = 0 \quad \text{otherwise}$$



g_0 **loses translation invariance** but the propagator g keeps the multiscale and long range properties of the initial model, **at least hierarchically**

But since the impurity is localized observ. localized at 0
depend on fields at 0, $\psi^\pm(0), \varphi^\pm \Rightarrow$ **1D problem** (AYH).

Illustration of (AYH970) remark: **1D problem**, (long range)
Main operators in the Lagrangian:

$$O_0(t) \stackrel{def}{=} \psi^+(t) \boldsymbol{\sigma} \psi^-(t) \cdot \boldsymbol{\tau} = \vec{A}(t) \cdot \boldsymbol{\tau}, \quad O_5(t) \stackrel{def}{=} \boldsymbol{\tau} \cdot \boldsymbol{\omega}$$

(in Grassmannian form) and

$\mathcal{L}_K$ on scale m is (with $\alpha_0 < 0, \alpha_5 = h \geq 0$ else 0).

$$\int e^{\mathcal{L}_K^{[\leq m]}(\psi^{[\leq m]})} d\psi = \int e^{-\int_0^\beta \sum_i \alpha_i^{[m]} O_i(t) dt} d\psi^{[0]} d\psi^{[1]} \dots d\psi^{[m+1]}$$

Set RG analysis via (Grassmannian) for $\text{Tr } e^{-\beta H_K}$

Key: **IF** $h = 0$ then $\mathcal{L}_K^{[m]}(t)$ is $\forall m$:

$$\alpha_0^{[m]} O_0(t) \cdot \boldsymbol{\tau} + \alpha_1^{[m]} O_1(t)$$

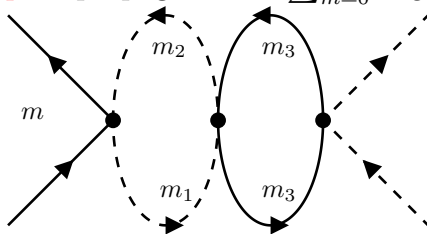
i.e. no new operators needed at any scale (exact recursion)

Scaling $O_0 =$ marginal, O_1 irrelevant, $O_5 =$ relevant

The RG consists in

1) Expand **perturbatively** $Z^{[>m]} = e^{V^{[m]}}$ via Feynman gr.
heavily using the hierarchical structure

2) **Decompose** propagators as $\sum_{m=0}^{-\infty} 2^m g_0(2^m(t-t'))$



3) **Recognize**: at $h = 0$ no new operators can arise besides

$$O_4 = \vec{A} \cdot \vec{h}, \quad O_5 = \boldsymbol{\sigma} \cdot \vec{h}, \quad O_6 = \vec{A} \cdot \vec{h} \boldsymbol{\sigma} \cdot \vec{h}, \quad O_7 = \vec{A}^2 \boldsymbol{\sigma} \cdot \vec{h},$$

3) **Recognize** that the result contains a few series that can be collected to form a sequence of **running couplings**

$$\boldsymbol{\alpha}^{[m]} = (\alpha_0^{[m]}, \alpha_1^{[m]}, \alpha_4^{[m]}, \alpha_5^{[m]}, \alpha_6^{[m]}, \alpha_7^{[m]}).$$

with only $\alpha_0^{[m]}, \alpha_1^{[m]} \neq 0$ if $h = 0$

4) Each is a convergent series in the initial couplings α_0, h , **if small enough** (BUT converg. radius **m dependent**)

5) **Recognize** that the $\boldsymbol{\alpha}^{[m]}$ satisfy a formal recursion

$$\boldsymbol{\alpha}^{[m]} = \Lambda \boldsymbol{\alpha}^{[m+1]} + \mathcal{B}(\boldsymbol{\alpha}^{[m+1]})$$

and $\mathcal{B}$ can be expressed as a “polynomial” with coefficients which are geometric series in $\boldsymbol{\alpha}^{[m+1]}$; $\Lambda = (1, \frac{1}{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \frac{1}{2})$.

Even **forgetting** convergence, **PT of no use**: marginal term grows (if $\lambda_0 < 0$) and generates growing (“relevant” terms)!

6) **Sum the geometric series** to obtain a **closed form** of $\mathcal{B}$.
After a natural change of variables $\alpha \longleftrightarrow \lambda$ at $h = 0$

$$\lambda'_0 = \frac{1}{C}(\lambda_0 + 3\lambda_0\lambda_1 - \lambda_0^2)$$

$$\lambda'_1 = \frac{1}{C}(\frac{1}{2}\lambda_1 + \frac{1}{8}\lambda_0^2),$$

$$C = 1 + \frac{3}{2}\lambda_0^2 + 9\lambda_1^2$$

Non perturbative: for $m \rightarrow -\infty$ (IR limit, $\beta = +\infty$, $T = 0$)

$\lambda^{[m]}, \alpha^{[m]}$ converge to **non trivial fixed point**

if $h = 0, \alpha_0 < 0$, **exactly computable**,

$$\lambda_0^* = -7.807257...10^{-1}, \lambda_1^* = 5.292875...10^{-2}$$

$$\lambda_0^* = -x \frac{1 + 5x}{1 - 4x}, \lambda_1^* = \frac{x}{3}, x = 7.807257...10^{-1},$$

with $4 - 19x - 22x^2 - 107x^3 = 0$, real root.

Susceptibility: new operators needed to close beta

$$O_4 = \vec{A} \cdot \vec{h}, \quad O_5 = \boldsymbol{\sigma} \cdot \vec{h}, \quad O_6 = \vec{A} \cdot \vec{h} \boldsymbol{\sigma} \cdot \vec{h}, \quad O_7 = \vec{A}^2 \boldsymbol{\sigma} \cdot \vec{h},$$

O_0, O_4, O_6 marginal, O_5 relevant, O_1, O_7 irrelevant

Calculating beta function: via Feynman graphs, after simplifications, a beta function with 36 coeff is found

From the flow of the α the partition function $Z(\beta, h)$ is computed and susceptibility

$$\chi(\beta, h) = \partial_h^2 \log Z(\beta, h)$$

follows as a function of h .

The beta function is a rational function defined by the ratio of two polynomials of degree 2.

$$C = 1 + \lambda_0^2 + \frac{1}{2}(\lambda_0 + \lambda_6)^2 + 9\lambda_1^2 + \frac{1}{2}\lambda_4^2 + \frac{1}{4}\lambda_5^2 + 9\lambda_7^2$$

$$\lambda'_0 = \frac{1}{C}(\lambda_0 - \lambda_0^2 + 3\lambda_0\lambda_1 - \lambda_0\lambda_6)$$

$$\lambda'_1 = \frac{1}{C}(\frac{1}{2}\lambda_1 + \frac{1}{8}\lambda_0^2 + \frac{1}{12}\lambda_0\lambda_6 + \frac{1}{24}\lambda_4^2 + \frac{1}{4}\lambda_5\lambda_7 + \frac{1}{24}\lambda_6^2)$$

$$\lambda'_4 = \frac{1}{C}(\lambda_4 + \frac{1}{2}\lambda_0\lambda_5 + 3\lambda_0\lambda_7 + 3\lambda_1\lambda_4 + \frac{1}{2}\lambda_5\lambda_6 + 3\lambda_6\lambda_7)$$

$$\lambda'_5 = \frac{1}{C}(2\lambda_5 + 2\lambda_0\lambda_4 + 36\lambda_1\lambda_7 + 2\lambda_4\lambda_6)$$

$$\lambda'_6 = \frac{1}{C}(\lambda_6 + \lambda_0\lambda_6 + 3\lambda_1\lambda_6 + \frac{1}{2}\lambda_4\lambda_5 + 3\lambda_4\lambda_7)$$

$$\lambda'_7 = \frac{1}{C}(\frac{1}{2}\lambda_7 + \frac{1}{12}\lambda_0\lambda_4 + \frac{1}{4}\lambda_1\lambda_5 + \frac{1}{12}\lambda_4\lambda_6)$$

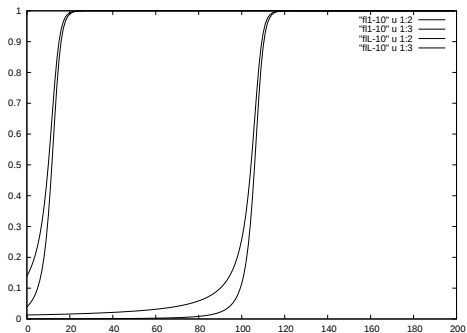


Fig.2: plot of $\frac{\lambda_i}{\lambda_i^*}$, $i = 0, 1$, as a function of $N_\beta = \log_2 \beta$, $\lambda_0 \equiv \alpha_0 = -0.1, -0.01$ respectively the left and the right pairs.

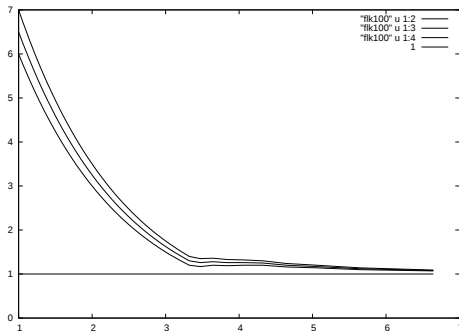


Fig.3: inflection point $n_0(\lambda_0)$: $n_0(\lambda_0) \cdot |\lambda_0|$ vs. $|\log_2 |\lambda_0||$: only data with 10% error (upper and lower curves) visual lines interpolate data

$$T_K = \text{const } e^{-c_0 \lambda_0^{-1}}$$

For $h \neq 0$ the flow leads to “high T fixed pt.” at scale $\propto 1/|\log h|$

The equation of state

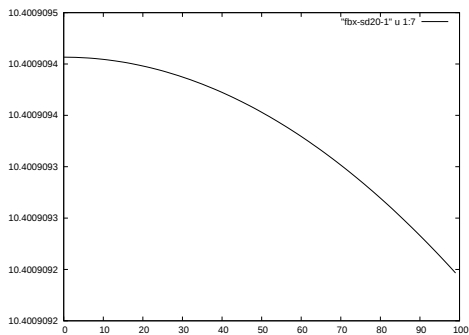


Fig.4: plot of $\chi(\beta, h)$ for $h \in [0, 10^{-6}]$ at $\lambda_0 = -0.3$ and $\beta = 2^{20}$ (so that the largest value for βh is ~ 1)

[1, 2, 4, 3, 5]

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