SRB & Boltzmann's distributions: hyperbolicity and ergodic hypothesis

Basic assumption ("chaotic hypothesis") refers to equilibrium and nonequilibrium phenomena alike: once a system undergoes chaotic motions it shows them in a maximal sense: $\Rightarrow$ to be intended, for the purpose of modeling its behavior, to be an "Anosov system".

Trivial to exhibit counterexamples: *e.g.*the horocyclic flow on a surface of constant < 0 curv, [1].

However keep in mind that Statistical Mechanics developed from (Boltzmann, Clausius, Maxwell) supposing microsc. motions periodic, [2, 3, 4],

Chaotic system admits (uncountably) many invariant distributions on their microscopic configurations; how do the ones relevant for Physics emerge?

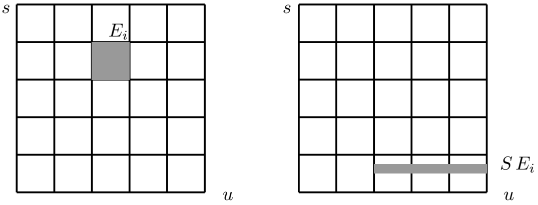
Boltzmann realized the problem in 1871, casting doubts on his early derivation (1868) of the microcanonical ensemble; resolved them essentially via the ergodic hypothesis.

Became soon dubious whether phase space volume invariance (Hamiltonian) is of any help: many attempts to connect micr. evolution with macr. properties via "coarse grained" representations have met difficulties (hyperbolic nature $\Rightarrow$ strong phase space cells in deformation.

In stationary states out of equilibrium volume invariance not even true. And the many invariant distr. are all (strongly) mixing $\Rightarrow$ difficult to see how the Boltzmann or SRB distr. could emerge as the ones relevant for Physics.

Physics: hyperbolicity, (restrict to maps for simplicity and assume chaotic hypothesis), is essential to establish a coherent notion of "coarse grain" descriptions.

Let (M, S) : be a smooth Anosov map S on the phase space M and let $\mathcal{E} = (E_1, \ldots, E_n)$ be a "Markov partition" of M



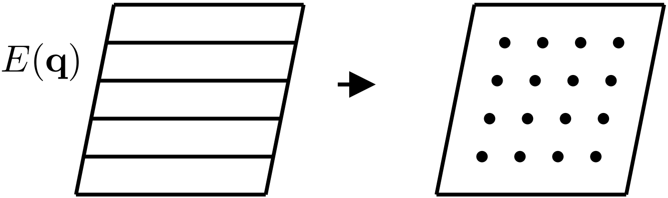
A $E_i \in \mathcal{E}$ transformed into SE_i s,t, shrinking part of boundary ends up exactly on boundary of some among $E_1, E_2, \ldots, E_n$. $\mathcal{E}$ chosen "**fine enough**" for the few interesting observables to be constant in each cell E_i.

$\Rightarrow$ symbolic represent. of motion: points $\to$ symbols $\boldsymbol{\sigma} = (\sigma_i)_{-\infty}^{\infty}$ with $M_{\sigma_i,\sigma_{i+1}} = 1$ determined by "transition matrix" $M_{\sigma,\sigma'} = 0, 1$ (transitive, *i.e.* $\exists n : (M^n)_{\sigma,\sigma'} > 0$).

"Ergodic hypothesis", classical, phase space discretized on regular lattice $R^{3N}_{\delta q} \times R^{3N}_{\delta p}$, of meshes $\delta q, \delta p$.

In confined Ham. sys. motion $\longleftrightarrow$ one cycle permutation of the (finite) set of points on given energy surface.

But conservative motions not required. However in general, *e.g.*out of equilibrium, evolution cannot be cyclic permutation. Points will be divided into *transient* and *recurrent*: and the *natural extension of the ergodic hypothesis* $\Rightarrow$ *all recurrent points are on a single cycle*.



Evolution drives to unstable manifolds of attracting set. Discrete attracting set points will be imagined located on a selected unstable manifold

After a transient, attractor will appear as union of unstable manifolds.

Probability of each element of $\mathcal{E}$ will simply be the number of points of the attractor in it, since motion is simply 1-cycle perm. of attractor points.

Statistics is *uniquely determined* and just uniform distribution on the attractor points.
Let σ_E = surface of attractor unstable surfaces (after discretization) inside E and N_E =number of attractor points on it. The condition of invariance of the attractor $\Rightarrow$ consistency on the N_E *i.e.* on the weights w_E.

If "cell" E_i receives points from cell $E_{i'}$ $(SE_{i'} \cap E_i \neq \emptyset)$ and the surface of the attractor is expanded by a factor $\Lambda(i')$ it must be

$$N_{E_i} = \sum_{i' \to i} \Lambda(i')^{-1} \frac{N_{E'_i}}{\sigma_{E_{i'}}} \sigma_{E_i} \;\Rightarrow\; w_i = \sum_{i' \to i} \Lambda(i')^{-1} w_{i'}$$

An equation that could be formulated without reference to a discretization: $\Rightarrow$ stationary distribution has density on unstable manifolds 1-quely determined by expansion rates.

Question: "what happened to the other (uncountably many) invariant distrib. of the continuum map (M, S)?"

Discretization has been done via a regular lattice. Using different discretizations, *e.g.*self similar discretizations of a fractal, the result would be different and other invariant distributions would arise, in the continuum limit, for the considered system (M, S) and $\neq$ SRB.

Relevance of the SRB distributions tightly related to structure of space-time: representability of natural laws via ODE or PDE is, ultimately, possible because a model of natural laws can be equivalently built on microscpically discrete and regular arrays or on Euclidean continua (a natural law itself).

This is quite explicitly invoked by Boltzmann, for instance in [5, p.169]:

Therefore if we wish to get a picture of the continuum in words, we first have to imagine a large, but finite number of particles with certain properties and investigate the behavior of the ensemble of such particles … one can then assert that they apply to a continuum, and in my opinion this is the only non-contradictory definition of a continuum with certain properties

Ergodicity does not solve the supposed conflict between micr. reversibility and macr. irreversibility.

However Boltzmann (and others, *e.g.*Thomson) had clearly reconciled the two and estimated the minimal time scale on which reversibility could be detected.

The idea of a rather general explanation of the nature of micr. motion, that inspired the founding fathers, is that a general explanation might imply general consequences directly accessible on human time scales.

A micr. property for rather arbitrary systems subject to basic laws of nature (*e.g.*classical mechanics) might be uninteresting in the case of systems with few particles but it might turn out to be a fundamental law for many particle systems.

The example of the second law of thermodynamics, derived by Boltzmann, Clausius, Maxwell, is a paradigmatic example, [2, 3, 4].

A general micr. viewpoint might be very useful to study general properties implied by symmetries:

thermodynamics reflects the Hamilt. symmetry of the basic equations, both from the viewpoint of periodic recurrence as in [2] or from the ensemble viewpoint in [6].

Onsager reciprocity reflects time reversal near equilibrium,

SRB theory is linked to the homogeneity of space time with respect to the Galilei group, as attempted above.

Other micr. symmetries can be reflected macr., once assumed the involved systems to follow a law of motion obeying a general common property.

an example, quite simple and nevertheless non trivial, is the "fluctuation theorem" which reflects the basic time reversal symmetry, [7], of dissipative systems.

Chaos: RPF, cones of functions Anosov maps/flows and manifestations of chaos

[8] n hard spheres (trapped in a channel and localized in boxes with holes): first example of $n > 2$ hard spheres translation invariant (> 0 Lyap. e ergodicity).

[9] Example ergodicity on T^2 under a central potential: original singularity improved to arbitrary power $r^{-\alpha}$-singularity but also smooth. Method of cones.

[10] Ergodicity in Hamiltonian Systems, a general formalization of "Sinai's method" with examples.

[11] First of correlations decays and detailed spectral properties studies for the RPF operator (hyperbolic): beyond 1-D maps and via the new idea of cones in spaces of functions ("observables"), rather than cones of tangent vectors, with "Hilbert metric"; leading to several examples.

[12] Sistems of ∞-many classical particles in R^3 subject to internal forces and to a reversible noise has Gibbs states as invariant distributions with short range stochastic interaction.

[13] A much needed analysis of what is (and/or should be) required from a simulation: the focus is on the possibility to answer the above questions, possibly via computer assisted strategies, but always with rigorous error bounds. Accompanied by the explicit rather unusual recognition that an analysis of the algorithmic complexity of the proposed methods would be required.."

[14] RPF operator for Anosov systems is studied in great detail on appropriate function spaces (called Banach spaces to delight the Physics minded readers). Also with attention to the coarse graining in the sense of Ulam.

[15] Conditionally invariant distributions on the interval have been studied and are a useful preliminary to a needed perturbation analysis (and def.) of metastability in DS

[16] I recommend several lectures on invariant measures and RPF operator. Mainly because of their attention to constructivity" (a constant in this list).

[17] Exponential decay for geodesic flows on surfaces with < 0 curvature (and Anosov contact flows) studied using the novel Dolgopyat's technique. Simpler, far reaching, alternative to traditional viewpoint of Markov part.

[18] Continuity and differentiability of spectral data of SRB distributions, based on the RPF operator.

[19] Escape rates and "metastability bewteen two quasi invariant states" Development of a perturbation analysis with respect to escape windows size; several examples.

[20] There is very original study of a chain of geodesic flows on < 0 curved surfaces with nearest neigbor coupling. It can be shown that suitably rescaling time and the energies of the fluxes the latter evolve via a diffusion process. Even attention to (lack of) dependence on regularization lattice.

[21] Zeta functions for Anosov fluxes and their perodic orbits: proof of Zetas meromorphy for C^∞ flows (Smale's conjecture) and accurate count of the number of periodic orbits.

[22, 23] Statistics of a DS with 2 time scales $1, \varepsilon$: all earlier methods merge to develop, combined with Varhadan-Dolgopyat martingale method, to study main interesting quantities in chaotic DS (CLT, Lyap., Exp. D.).

[24] Study of return to equilibrium: in ∞-quantum systems (via pairings and Hilbert metrics). Quantum chaos is not supposed to be any different from classical but there are not too many works.

[25] For instance Quantum System in Contact with a Thermal Environment: Rigorous Treatment of a Simple Model. QLE: heavy particle interact with a 1D quantum harmonic chain. Key feature: results depend on initial state; preliminarily conditions are found for FKM (conjectured) analogue can hold.

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