"On some integrable systems: Brownian motion, normal forms and zeros of polynomials"

Among the first works of Carlo was integration of Hamiltonian

$$H_n(\vec{p},\vec{x}) = \sum_{i=1}^{n} \frac{1}{2}p_i^2 + \frac{1}{2}\sum_{i=1}^{n} \omega^2 x_i^2 + \frac{1}{2}\sum_{i\neq j}^{1,n} g^2 \frac{1}{(x_i - x_j)^2} \qquad (*)$$

for $n = 3$, following early (quantum) results of Calogero, [3, 2]:

Besides integrating the classical system also the scattering matrix of the three bodies was calculated and found equal to that of a system of impenetrable points (1970),[10].

H_n: Essentially self-adjoint on $\mathcal{D}_{\lambda,\xi} \subset L_2(R^n)$

$$\prod_{i=1}^{n}\prod_{j=i+1}^{n} (x_i - x_j)^{\lambda} P(x_1,\ldots,x_n) e^{-\xi\sum_{i=1}^{n} x_i^2} \qquad (**)$$

with $\lambda = \mathbf{\frac{1}{2}(1 + (1 + \frac{4g^2}{\hbar^2})^{\frac{1}{2}})}$, if $\lambda > 3$, $\xi = \frac{\omega}{2\hbar}$ and 0 unless $x_1 < x_2 < \ldots < x_n$, P a symmetric polynomial in x_i.

Calogero then computed the full spectrum of the n body system with quadratic $\frac{1}{2}\omega^2 m\sum_{i<j}(x_i-x_j)^2$ determining the eigenfunctions (1971),[4].

Marchioro & G.: Eigenvectors for (*): same formula **(**)** with polynomials recursively generated.

Completeness: reduced to Hermite's polynomials completeness.

The H_n eigenvalues are ($k_1,\ldots,k_n$ integers mod permutations).

$$\varepsilon_{k_1,\ldots,k_n}=\omega\hbar\sum_{i=1}^{n}(k_i+\frac{1}{2})+\omega\hbar\lambda\frac{n(n-1)}{2}$$

Here H_n is restricted to $\cup_P\mathcal{D}^P_{\lambda,\xi}$ where $\mathcal{D}^P_{\xi,\lambda}$= image of $\mathcal{D}_{\lambda,\xi}$ under map $(x_1,\ldots,x_n)\to(x_{P_1},\ldots,x_{P_n})$, multiplicity is $n!$.
Partition function: $Z_n=\mathrm{Tr}e^{-\beta H_n}$: and (formally)

$$\lim_{\hbar\to0}(2\pi\hbar)^n Z_n\equiv\int e^{-\frac{\beta}{2}\sum_{i=1}^n p_i^2-\frac{\beta}{2}\sum_{i<j}\frac{g^2}{(x_i-x_j)^2}-\frac{\beta}{2}\omega^2\sum_{i=1}^n q_i^2}d\vec{p}d\vec{q}$$

$$=e^{-\beta\omega g\frac{n(n-1)}{2}}(\frac{2\pi}{\beta\omega})^n$$

Z_n can be directly computed from spectrum $\Rightarrow$ classical limit

$$\int e^{-\frac{\beta}{2}\sum_{i=1}^{n} p_i^2 - \frac{\beta}{2}\sum_{i<j}\frac{g^2}{(x_i-x_j)^2} - \frac{\beta}{2}\omega^2\sum_{i=1}^{n} q_i^2} d\vec{p}d\vec{q} = e^{-\beta\omega g\frac{n(n-1)}{2}}\left(\frac{2\pi}{\beta\omega}\right)^n$$

This formula is another contribution of Marchioro[9]: it is non trivial shown by its non inclusion in Gradshtein-Ryzhik tables.

Apparently after writing to Carlo that it would be included
it was not.

I think that the referees had trouble (still have?) understanding the delicate analysis of the limit as $\hbar \to 0$.

Nevertheless the formula has had considerable influence in the literature. I had the chance of collaborating with Carlo in its development.
A sketch of the logic on the formula.
Feynman-Kac for the kernel of $e^{-\beta H_n}$:

formally:

$$K_\beta(\vec{x},\vec{y}) = \int (\prod_{i=1}^{n} P_{x_i,y_i}(d\omega_i)) e^{-V_\beta(\boldsymbol{\omega})} \leq K_\beta^{g=0}$$

$$V_\beta(\boldsymbol{\omega}) \stackrel{def}{=} \frac{g^2}{2} \sum_{i\neq j} \int_0^\beta \frac{d\tau}{(\omega_i(\tau) - \omega_j(\tau))^2} + \frac{\omega^2}{2} \sum_{i=1}^{n} \int_0^\beta \omega_i(\tau)^2 d\tau$$

K_β is the kernel of a semigroup but is it that of $e^{-\beta H_n}$?
$n = 2$ would be sufficient.

Lemma $\to$ theorem: *Strong L_2 convergence, if $\lambda > 3$*:

$$\lim_{\beta\to 0} \int d\vec{y} \int P_{\vec{x},\vec{y}}(d\boldsymbol{\omega}) e^{-\frac{1}{\beta}(V_\beta(\boldsymbol{\omega})-1)}$$

$$= -\frac{g^2}{2} \sum_{i<j} \frac{1}{(x_i - x_j)^2} - \frac{\omega^2}{2} \sum_i x_i^2$$

and K_β is strongly continuous $\frac{1}{2}$-group $\equiv e^{-\beta H_n}$ on L_2.

Then the inequality (F-K formula)

$$K_\beta(\vec{x},\vec{y}) \le K_\beta(\vec{x},\vec{y})_{g=0} \stackrel{def}{=} K^0_\beta(\vec{x},\vec{y})$$

and K^0_β is well known: at $\vec{x}=\vec{y}$ (needed for the trace) it is

$$K^0_\beta(\vec{x},\vec{x}) = \Big(\frac{\omega\hbar}{\pi(1-e^{-2\beta\omega\hbar})}\Big)^{\frac{n}{2}} e^{-n\frac{\beta\omega\hbar}{2}} \prod_{i=1}^n e^{-\frac{\omega}{\hbar}\frac{1-e^{-\beta\omega\hbar}}{1+2e^{-\beta\omega\hbar}}\, x_i^2}$$

Yields *a priori* bounds leading directly to

$$\lim_{\hbar\to 0}(2\pi\hbar)^n \mathrm{Tr}\, e^{-\beta H_n}$$
$$= (\frac{2\pi}{\beta})^{\frac{n}{2}} \int d\vec{x} e^{-\frac{1}{2}\sum_{i=1}^n \omega^2 x_i^2 + \frac{1}{2}\sum_{i\neq j}^{1,n} g^2 \frac{1}{(x_i-x_j)^2}}$$

An important remark was that the same result would follow if the Hamiltonian was integrable and admitted, in each of the $n!$ sectors of $L_2(R^n)$ canonical action angle variables

$$(A_1,\ldots,A_n,\varphi_1,\ldots,\varphi_n),\ A_j\ge 0\ \sim\ (\vec p,\vec q)$$

and with normal form presented as integrability conjecture:

$$\widetilde H(\vec A,\boldsymbol\phi)=\sum_{k=1}^{n}k\,\omega\,A_k+\frac{\omega\,g}{2}\,n(n-1)$$

By symplectic invariance THEN the integral would be

$$\int d\vec A d\boldsymbol\phi e^{-\beta\sum_{k=1}^{n}k\,\omega\,A_k+\frac{\beta\omega\,g}{2}\,n(n-1)}$$

I learnt about the Lax pairs at a workshop and insisted with Moser about H_n should be integrable. I tried to find the pair : failing.

But Moser few weeks later discovered H. equations with $\omega=0$ could be written in terms of matrices

$$M_{ij}=\delta_{ij}(p_i-\omega^2x_i)+\frac{g^2\sqrt{-1}}{x_i-x_j},\qquad N_{ij}=\delta_{i\ne j}\frac{1}{(x_i-x_j)^2}$$

and become $\dot M=\sqrt{-1}\,[M,N]$: which implies n eigenvalues of the matrix M are n consts of motion, indep. & in involution.

Shortly later Adler realized integrability of Calogero's model [11, 1].
Among other results Moser was able to prove Marchioro's conjecture about the n-particles scattering.

This proved integrability but not yet the integrability conjecture: which was proved (much) later by Françoise.[6]

Recently, with Françoise and Garrido, a series of papers was then started (actually 2!) to analyze properties of integrable systems starting from the most elementary, with long run aim at the Calogero-Marchioro-Moser classical Hamiltonian.

Here I continue towards discussing some remarkable properties that arose, for the pendulum and the Poinsot motions, and that I think might be the expression of interesting structures

However we remain still far from getting close to the CMM-system or to other systems like the Kowalevskaia gyroscope or the Toda lattice.

Hamiltonian of a solid with inertia I_1, I_2, I_3, in Deprit's canonical coord. $(K_z, A, B, \gamma, \varphi, \beta)$, is

$$\widetilde{H}(K_z, A, B, \gamma, \varphi, \beta) = \frac{1}{2}\frac{B^2}{I_3} + \frac{1}{2}\Big(\frac{\cos^2\beta}{I_1} + \frac{\sin^2\beta}{I_2}\Big)\,(A^2 - B^2)$$

with $B = 0, \beta = 0$ the unstable uniform rotation if $I_3 < I_1 < I_2$.

$$J = I_1,\ \widetilde{J} = I_3^{-1} - I_2^{-1},\ J_{31} = I_3^{-1} - I_1^{-1},\ r^2 \stackrel{def}{=} \frac{J_{12}^{-1}}{\widetilde{J}^{-1}},$$

Theorem: *Variables p, q can be constructed so that the Hamiltonian takes the form*

$$\frac{A^2}{2I_2} - \frac{xA^2r^2}{2J_{31}}\mathcal{H}(x, r^2)$$

where $x = pq$ so that evolution is

$$p \to pe^{g(x,r^2)t},\ q \to qe^{-g(x,r^2)t}, \qquad g(x) = \partial_x\mathcal{H}(x)$$

[7, 8]

The normal form $\mathcal{H}(x)$ is analytic in x (small) with coefficients which are polynomials in r^2:

$$\mathcal{H}(x) = 4x - 2(1+r^2)x^2 - (-1+r^2)^2x^3 - \frac{5(1+r^2)(-1+r^2)^2}{4}x^4$$
$$- \frac{3(-1+r^2)^2}{16}(11+10r^2+11r^4)x^5 - \frac{7(-1+r^2)^2(1+r^2)}{16}(9-2r^2+9r^4)x^6$$
$$- \frac{(-1+r^2)^2}{64}(527+332r^2+330r^4+332r^6+527r^8)x^7$$
$$- \frac{9(1+r^2)(-1+r^2)^2}{512}(1043-548r^2+1058r^4-548r^6+1043r^8)x^8 + \dots$$

The coefficients can be expressed via combinatorial coefficients and empirically up to order x^{15} have all roots on the unit circle.

Attempts to show that the polynomials (at least up to order 15) are of Lee-Yang type *i.e.* that can be written as

$$Q(z^2) = \sum_{\sigma_1,\dots,\sigma_n=\pm 1} e^{\sum_{i,j} J_{ij}\left(\frac{\sigma_i+\sigma_j}{2}\right)^2} z^{\sum_i \sigma_i}$$

with $J_{ij} > 0$. Then conjecture implied by Lee-Yang's theorem

In this connection there is a theorem

Theorem: (Chen, 1995) Polynomials with symmetric, positive and monotonic coefficients have all roots on the unit circle.[5]

Is it true that such polynomials are Lee-Yang polynomials?

The latter is a conjecture that can be checked rigorously for $n \le 5$ and we found some preliminary numerical evidence that it should also work at least for $n \le 8$.

The proof for $n = 4$ emerged from a discussion with A. Giuliani; he also has a proof solving the case $n = 5$ (and found the reference to Chen's theorem).

The polynomials in r^2 appearing as coefficients of x^k in the normal form are sums of products of polynomials P_{n_j} of degrees which add up to $2(k-1) = \sum_j n_j$: an expression for related polynomials can be found in terms of a "tree expansion": hence have "quite explicit combinatorial expressions" (useful for the conjecture?).

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