

## Reversibility, Irreversibility, Friction and nonequilibrium ensembles in N-S equations

**Question:** can the phenomenological notion of friction be represented in **alternative** ways?

**Related (?) Q.** is it possible to set up a theory of statistical ensembles, and their equivalence, **extending** to stationary non-equilibria the ideas behind the canonical and microcanonical ensembles.

**Guide:** *a fundamental symmetry like “time reversal” cannot be “spontaneously broken”*

*Therefore even the stationary states of dissipative systems ought to be describable **via time reversible equations.***

*It will be better to specialize on a paradigmatic example, the **NS fluid in a  $2\pi$ -periodic box, 2/3-D.  $R \equiv \frac{1}{\nu}$  be Reynolds #.***

$$NS_{irr}: \dot{u}_\alpha = -(\vec{u} \cdot \partial)u_\alpha - \partial_\alpha p + \frac{1}{R}\Delta u_\alpha + F_\alpha, \quad \partial_\alpha u_\alpha = 0$$

$$Velocity: \vec{u}(x) = \sum_{\vec{k} \neq \vec{0}} u_{\vec{k}} \frac{\mathbf{k}^\perp}{|\mathbf{k}|} e^{i\mathbf{k} \cdot \mathbf{x}},$$

$$NS_{2,irr}: \dot{u}_{\mathbf{k}} = - \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} \frac{(\mathbf{k}_1^\perp \cdot \mathbf{k}_2)(\mathbf{k}_2 \cdot \mathbf{k}_1)}{|\mathbf{k}_1||\mathbf{k}_2||\mathbf{k}|} u_{\mathbf{k}_1} u_{\mathbf{k}_2} - \nu \mathbf{k}^2 u_{\mathbf{k}} + F_{\mathbf{k}}$$

Although the 2D-NS admit general smooth solution *it is convenient to imagine it* (aiming at 3D-NS) as truncated at  $|\mathbf{k}| \leq N$ . The UV-cut-off  $N$  will be fixed for a while.

The 2D NS become  $4N(N+1)$  ODE's, on phase space  $M_N$ . (In 3D  $O(N^3)$ ).

$Iu_\alpha = -u_\alpha$  does *not* imply  $IS_t = S_{-t}I$ ,  $\Rightarrow$ : these are irreversible equations.

Let  $u$  be an initial state: then  $t \rightarrow S_t u$  evolves and generates a stationary state on  $M_N$  which, *aside exceptions collected in a 0-volume in  $M_N$ , is supposed unique, for simplicity*. Let  $\mu_R(du)$  be its PDF.

Stationary PDFs generalize equilibrium ones: thus collection  $\mathcal{E}^c$  of the  $\mu_R(du)$  will be called an **ensemble of nonequil. distrib.** for  $NS_{irr}$ .

Hence average energy  $E_R$ , average dissipation  $En_R$ , (local) Lyapunov spectra  $\mathcal{L}_R \dots$ , will be defined, e.g.:

$$E_R = \int_{M_N} \mu_R(du) \|u\|_2^2, \quad En_R = \int_{M_N} \mu_R(du) \|\mathbf{k}u\|_2^2$$

Consider the *new equation*,  $NS_{rev}$ :

$$\dot{\mathbf{u}}_{\mathbf{k}} = - \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} \frac{(\mathbf{k}_1^\perp \cdot \mathbf{k}_2)(\mathbf{k}_2 \cdot \mathbf{k}_1)}{|\mathbf{k}_1| |\mathbf{k}_2| |\mathbf{k}|} \mathbf{u}_{\mathbf{k}_1} \mathbf{u}_{\mathbf{k}_2} - \alpha(\mathbf{u}) \mathbf{k}^2 \mathbf{u}_{\mathbf{k}} + F_{\mathbf{k}}$$

with  $\alpha$  s. that  $En(u) = \|\mathbf{k}u\|_2^2$  is exact constant of motion:

$$\alpha(u) = \frac{\sum_{\mathbf{k}} \mathbf{k}^2 \operatorname{Re}(F_{-\mathbf{k}} u_{\mathbf{k}})}{\sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2} \quad \text{if } D = 2$$

The new equation is reversible:  $IS_t u = S_{-t} I u$  (as  $\alpha$  is odd).

So  $\alpha$  is “*reversible friction*”; (if  $D = 3$  slightly different)

This can be thought as a “*thermostat*” acting on the system and it should (?) have same effect as constant friction.

The evolution with  $NS_{rev}$  generates a family of stationary distributions on phase space:  $\mu_{E_n}^{mc}$  *parameterized by the constant value of the dissipation*  $En = \sum_{\mathbf{k}} |\mathbf{k}|^2 |u_{\mathbf{k}}|^2$ .

Denote  $\mathcal{E}^{mc}$  such collection of stationary PDFs.

The  $\alpha(u)$  in  $NS_{rev}$  will fluctuate strongly if the Reynolds number is large and it will “self-average” to a constant  $\nu$  thus “homogenizing” the equation and turning it into the  $NS_{irr}$  with friction  $\nu$ . A *first* more precise statement:

*The averages of large scale observables will show the same statistical properties, as  $R \rightarrow \infty$ , in the  $NS_{irr}$  and in the  $NS_{rev}$  equations under the correspondence*

$$\mu_R^c \longleftrightarrow \mu_{E_n}^{mc} \quad \text{if} \quad \mu_R^{mc}(En(u)) = En$$

By *large scale observables* it is simply meant “observables depending on the Fourier’s components  $u_{\mathbf{k}}$  with  $|\mathbf{k}| < K$  with some fixed  $K$ ”. And given  $K$  and such an observable it should be

$$\mu_{\mathbf{R}}^c(O) = \mu_{E_n}^{mc}(O)(1 + o(1/R)) \quad \text{if}$$
$$\mu_{E_n}^{mc}(\alpha) = \frac{1}{\mathbf{R}} \quad \text{or} \quad \mu_{\mathbf{R}}^c(\|\mathbf{k}\mathbf{u}\|^2) = \mathbf{E}_n$$

Recalls *canon.-microcan. equivalence*:  $\nu = \frac{1}{\mathbf{R}}$  plays the role of the canonical temperature ( $\beta$ ) and  $E_n$  that of microcanonical energy.

*Is the limit  $R \rightarrow \infty$ , or strong chaos, the analogue of the thermodynamic limit?*

The conjecture presented here is **no** for equations, like NS, which *follow from fundamental microscopic dynamics*.

## < 0 Examples:

(1) (highly) truncated NS equations ( $N < \infty$ ), [1],

(2) NS with Ekman friction, [2, 3],

(3) Lorenz96 model, [4],

(4) Turbulence shell model, (GOY), [5]

where the equivalence is possibly achieved *only in the limit of infinite forcing,  $R \rightarrow \infty$ .*

## > 0 Examples:

(1) The NS-equation: which can be derived from first principles. For instance for  $NS_{irr}$  (derived by Maxwell from *molecular motion*, [6]) it is natural to think that there **should be no condition for strong chaos.**

The microscopic motion is *always strongly chaotic* and the chaoticity condition should be always fulfilled even when **motion appears laminar.**

To pursue this suggestion consider the truncated  $NS_{rev/irr}$  equations at momentum  $N$ : in dimension 2 or 3. Then

The large scale observables, depending on the modes  $|\mathbf{k}| < K$ , have the same statistics in corresponding PDFs in  $\mathcal{E}^c$  and  $\mathcal{E}^{mc}$  in the limit  $N \rightarrow \infty$  for all  $R$  or  $En$

The analogy with Equilibrium Stat. Mech. is clear:

- (a) The (**necessary** if  $D = 3$ ) cut-off  $N$  plays the role of the finite volume container
- (b) the short scale cut-off  $K$  restricts attention to local observables
- c) the Reynolds number  $R$  plays the role of inverse temperature  $\beta$  and the dissipation  $En$  the role of the microcanonical energy.

**Then**

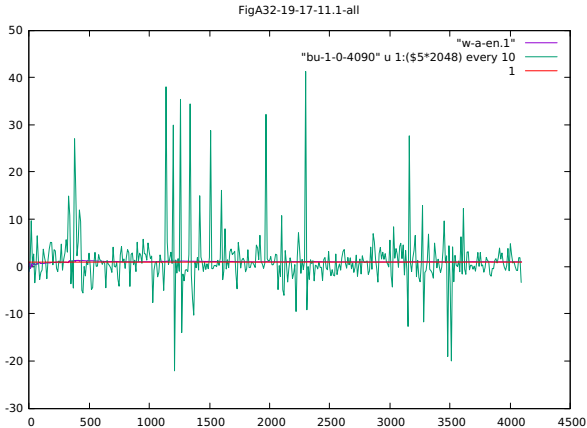
$$\lim_{N \rightarrow \infty} \mu_{En}^{mc}(O) = \lim_{N \rightarrow \infty} \mu_R^c(O)$$

for  $O(u)$  depending on  $u_{\mathbf{k}}$  with  $|\mathbf{k}| < K$  and under the equivalence relation (*i.e.*  $\mu_{En}^{mc}(\alpha) = \frac{1}{R}$ ): of course the larger  $K$  the larger  $N$  needs to be, **just as in equilibrium Stat. Mech.**

The above equivalence conjectures suggest way to perform measurements on real fluids which reveal the “hidden” reversibility of the motions.

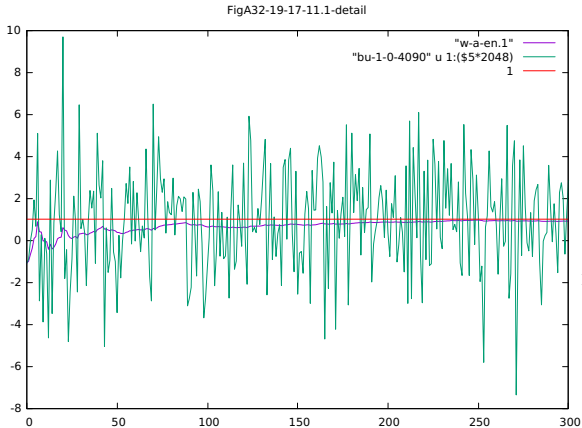
At this point it is convenient to pause and show a few results of simulations which begin to test the equivalence proposal.





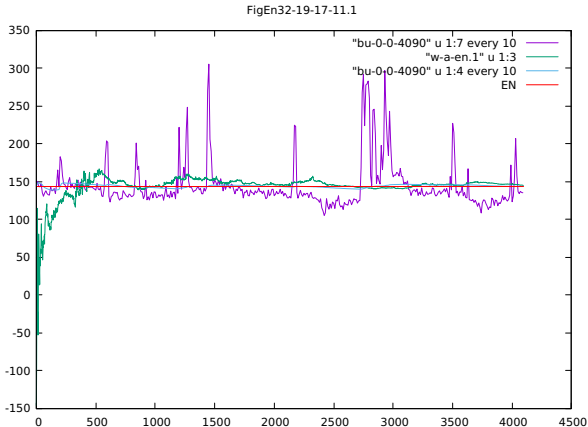
FigA32-19-17-11.1-all

Fig.1: The running average of the reversible friction  $R\alpha(u) \equiv R \frac{2\text{Re}(f_{-\mathbf{k}_0} u_{\mathbf{k}_0}) \mathbf{k}_0^2}{\sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2}$ , superposed to the **conjectured value 1** and to the fluctuating values  $R\alpha(u)$ : Evolution  $NS_{rev}$ , **R=2048**, 224 modes,  $\text{Lyap.} \simeq 2$ ,  $x$ -axis unit  $2^{19}$



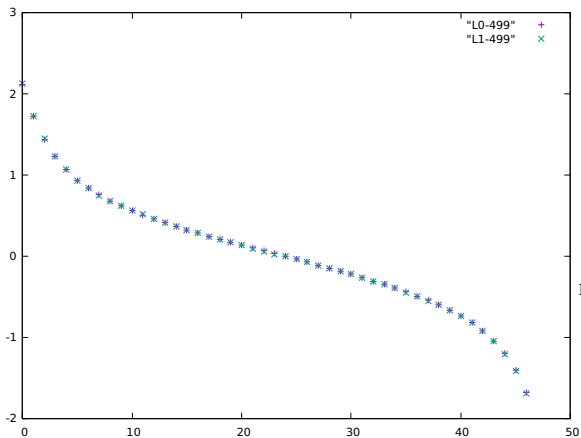
FigA32-19-17-11.1-detail

Fig.1-detail: The running average of the reversible friction  $R\alpha(u) \equiv R \frac{2\text{Re}(f_{-\mathbf{k}_0} u_{\mathbf{k}_0}) \mathbf{k}_0^2}{\sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2}$ , superposed to the **conjectured value 1** and to the fluctuating values  $R\alpha(u)$ : Evolution  $NS_{rev}$ , **R=2048**, 224 modes,  $\text{Lyap.} \simeq 2$ , x unit  $2^{19}$



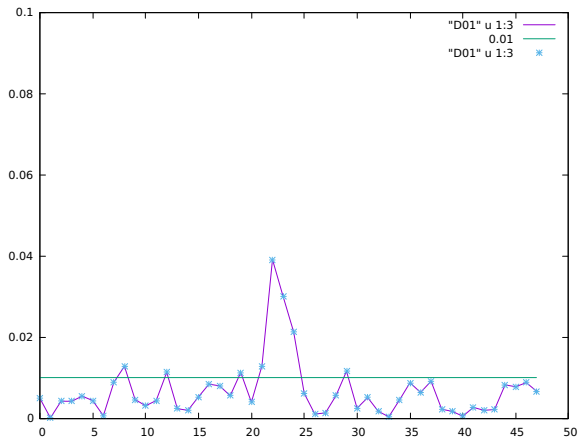
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Fig.2: Running average of  $R \sum_{\mathbf{k}} F_{-\mathbf{k}} |u_{\mathbf{k}}|$  (**dark green**)  $NS_{rev}$  converges to the average of  $\sum_{\mathbf{k}} \mathbf{k}^2 |u_{\mathbf{k}}|^2$  (straight **red** line)  
**green** line = running average of  $\sum_{\mathbf{k}} \mathbf{k}^2 |u_{\mathbf{k}}|^2$  in  $NS_{irr}$   
 large **fluctuations** are those of  $\sum_{\mathbf{k}} |u_{\mathbf{k}}|^2$ ,  $NS_{irr}$ : **R=2048**.



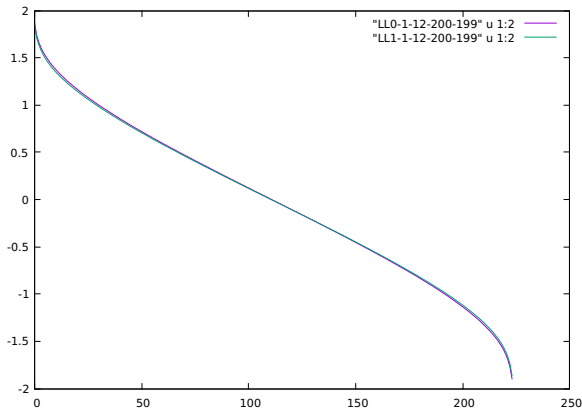
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Fig.3: The (**local**) Lyapunov spectra for 48 modes truncation: reversible and irreversible. And almost pairing,  $R=2048$ .



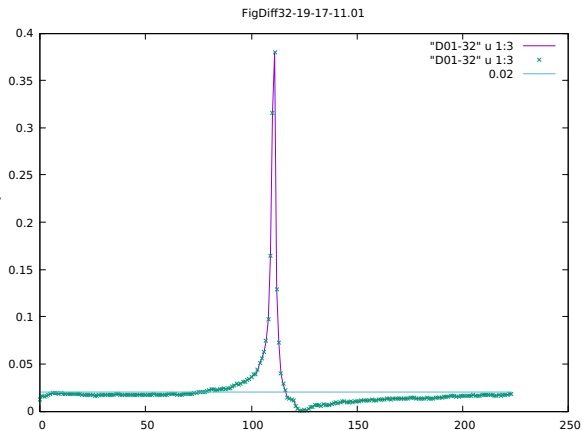
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Fig.4: Relative difference between (local) Lyapunov exponents in the previous Fig.  $R=2048$ , 48 modes.



FigL32-19-17-11.01

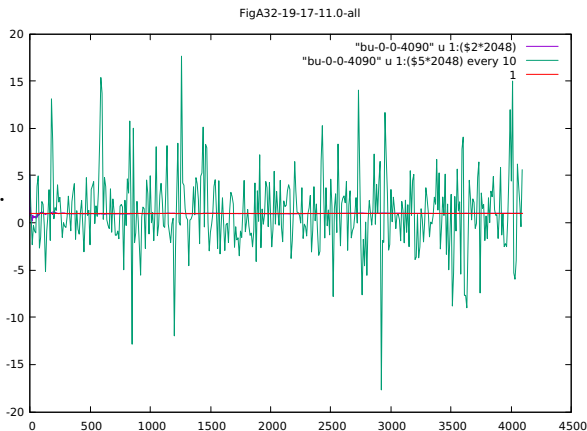
Fig.5: Local **Lyapunov spectra** in a  $15 \times 15$  truncation for the NS2D with viscosity and reversible viscosity (captions ending **respectively in 0 or 1**), interpolated by lines,  $R = 2048$ .  $\sim 2200$  are loc. ( $2^{13}$  steps) spectra evaluated, every  $2^{19}$  int. steps (running average).



FigDiff32-19-17-11.01

Fig.6: Relative difference between (local) Lyapunov exponents in the previous Fig.  $R=2048$ , 48 modes.

The following Fig.7 (similar to Fig.1 but w.  $NS_{rev}$ ):



FigA32-19-17-11.0-all

Fig.7: The running average of the reversible friction  $R\alpha(u)$  as seen by  $NS_{irr}$ , superposed to the conjectured value 1 and to the fluctuating values  $R\alpha(u)$  also in the irreversible  $NS_{irr}$ . Data correspond to those in Fig.1 above which came from  $NS_{rev}$ .



Suggests (from the theory of Anosov systems):

(1) **Test** the “Fluctuation Relation” in the linearized **irreversible** evolution of the Jacobian: if  $p = \frac{1}{\tau} \int_0^\tau \frac{\sigma(t)}{\langle \sigma \rangle} dt$  is finite time average of the **reversible friction** ( $\sigma(u) = -\sum_{\mathbf{k}} \partial_{\mathbf{k}}(\dot{u}_{\mathbf{k}})_{rev}$ ) then

$$\frac{P_{srb}(p)}{P_{srb}(-p)} = e^{\tau p \langle \sigma \rangle} \quad (\text{as large deviat. as } \tau \rightarrow \infty)$$

a “*reversibility test on the irreversible flow*”.

(2) **If FR is respected** then a new ensemble  $\mathcal{E}^{st}$  can be introduced consisting in the stationary states for the  $NS_{st}$

$$\dot{u}_\alpha = -(\vec{u} \cdot \boldsymbol{\partial})u_\alpha - \partial_\alpha p + \nu(u)\Delta u_\alpha + F_\alpha, \quad \partial_\alpha u_\alpha = 0$$

where  $\nu(u)$  is a gaussian process **uncorrelated in time** but with **average**  $\langle \nu \rangle = \frac{1}{R}$  and PDF respecting the FR (*i.e.* **dispersion equal to the average**)

Anosov systems play the role, in **chaotic dynamics**, of the **harmonic oscillators** in ordered dynamics. They are the paradigm of Chaos.

This idea rests on the work of **Sinai** (on Anosov sys.), **Ruelle, Bowen** (on Axioms A sys.), [7, 8, 9]

Accent on Anosov sys. has led to the

**Chaotic hypothesis:** *A chaotic evolution takes place on a smooth surface  $\mathcal{A}$ , “attracting surface”, contained in phase space, and on  $\mathcal{A}$  the maps  $S$  (or the flow  $S_t$ ) is an Anosov map (or flow).*

A strict, general, **heuristic**, interpretation of original ideas on turbulence phenomena, [9], see [10, endnote 18], [11, 12], [13].

It is dismissed (by many) with arguments like (1999)

'More recently Gallavotti and Cohen have emphasized the "nice" properties of Anosov systems. Rather than finding realistic Anosov examples they have instead promoted their "Chaotic Hypothesis": if a system behaved "like" a [wildly unphysical but well-understood] time reversible Anosov system there would be simple and appealing consequences, of exactly the kind mentioned above. Whether or not speculations concerning such hypothetical Anosov systems are an aid or a hindrance to understanding seems to be an aesthetic question.'

While giving up evaluating the statement I stress that Statistical Mechanics, after Clausius, Boltzmann and Maxwell was a simple and appealing consequence of the "[wildly unphysical but well-understood]" periodicity of motions of atoms in a gas, [14].

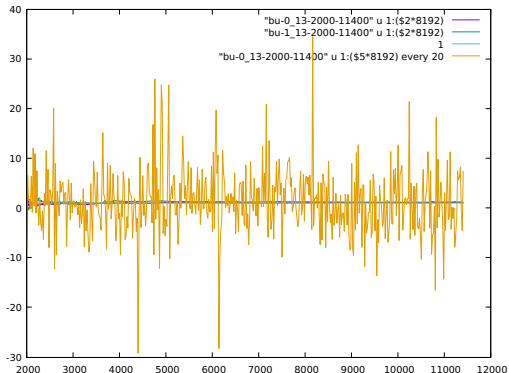
More elaborate tests are under way:

(a) **moments** of large scale observables rev & irrev

(b) study (local) **Lyapunov exponents of other matrices** instead of the Jacobian

(c) there is evidence that already with 224 modes the dimension of the attracting surface is **lower** than the phase space dimension:  $\Rightarrow$  **Fluct. Rel. with slope  $< 1$**  (Axiom C ?), [12, 11].

Other matrices can have exponents much larger hence (local) L. exp. may be easier to compute. Only preliminary results are available.



FigA.0-13-2000-11400-13

Higher  $R = 8192$ , 224 modes: running averages of  $R\alpha(u)$  for  $NS_{irr}$  &  $NS_{rev}$ , (predicted 1) and fluctuations for the  $NS_{irr}$ . Time recorded every  $4\lambda_{max}^{-1}$ .

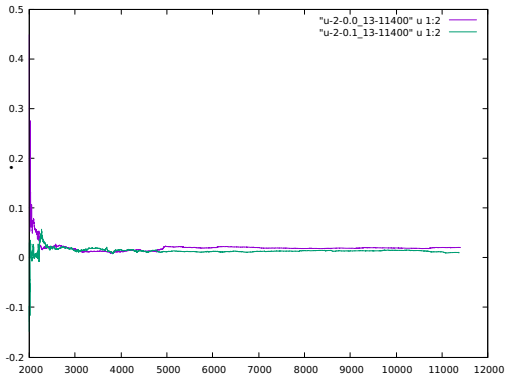


Fig20-0/1-19-17-13

Running averages **rev/irr** of the  $|u_{20}|^2$  component,  
 $R = 8192$ , 224 modes.

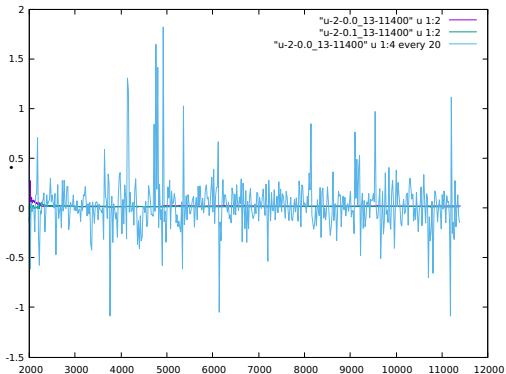
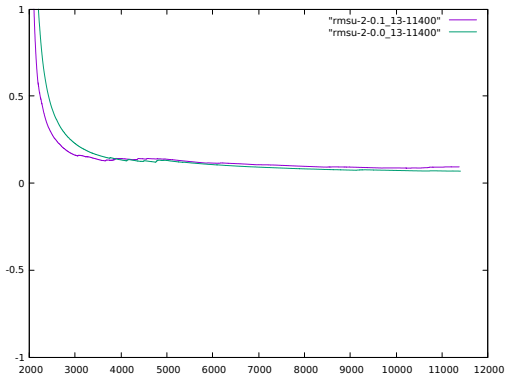


Fig20-0/1-19-17-13

Same running averages **rev/irr** of the  $|u_{20}|^2$  component,  $R = 8192$ , plus the fluctuations in the irr case, 224 modes.



FIGrmsu20-0/1-19-17-13

RMS for the above  $|u_{20}|^2$  **rev/irr**,  $R = 8192$ , 224 modes



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