

Statistical ensembles out of equilibrium: turbulence

Equilibrium states $\Rightarrow \mathcal{N} = \rho V$ particles in volume V and interaction potential $U(\mathbf{q}) \Rightarrow$ probability distributions determining average values of many observables

- 1) *i.e.* local observables $O \in \mathcal{O}_{loc}$: $O(\mathbf{p}, \mathbf{q})$, depend on $q_i \in \mathbf{q}$ located in regions $\Lambda \subset V$.
- 2) distributions **depend** on equations of motion.

Which among the invariant prob. distr. is the correct one?
For isolated systems **Ergodic Hypothesis** (EH) provides (a) solution: for a.a. data $\mathbf{u} = (\mathbf{p}, \mathbf{q})$

$$\mu_E(d\mathbf{p}d\mathbf{q}) = \frac{1}{Z} \delta(H_V(\mathbf{p}, \mathbf{q})) d\mathbf{p}d\mathbf{q}$$

As the energy $E = eV$ varies the distributions are collected in $\mathcal{E}_E^{mc,V}$, **microcanonical ensemble**.

Why? data are **always** generated randomly with a **unknown** distribution which however is (**tacitly ?**) assumed of the form $\rho(\mathbf{u})d\mathbf{u}$.

If system is chaotic (**e.g. hyperbolic**) it is a theorem that a.a data \mathbf{u} evolve visiting sets with well defined frequency **independent** of the unknown distribution for the data generation, called the **SRB distribution**.

This asymptotic behavior only depends on the hyperbolicity of the motion **on phase space** or, in the case of dissipative evolution, **on the attracting set**.

The two remarks contain the essence of **Ruelle's proposal**:

“the initial data are random with distr, $\rho(\mathbf{u})d\mathbf{u}$ (unknown but absolutely cont.) and motions are “generically” chaotic so that the statistics of the motions is uniquely determined as the SRB distribution”

If account is taken that **only few observables** are physically interesting then **other distributions** might provide the **same averages** for the interesting observables, particularly in the case of macroscopic systems, and can be collected in other “ensembles” \mathcal{E}_β^V . For instance the canonical distrib.

$$\mu_\beta^{c,V}(d\mathbf{p}d\mathbf{q}) = \frac{1}{Z} e^{-\beta H_V(\mathbf{p},\mathbf{q})} d\mathbf{p}d\mathbf{q}$$

The apparent resulting **ambiguity** is solved, in equilibrium, by the **equivalence** between distrib. in $\mathcal{E}_E^{mc,V}$ and $\mathcal{E}_\beta^{c,V}$:
“canonical distribution $\mu_\beta^V \in \mathcal{E}^c$ is equivalent to the microcanonical $\tilde{\mu}_E^V \in \mathcal{E}^{mc}$ if β, E are s.t.

$$\mu_\beta^V(H_V(\mathbf{p}, \mathbf{q})) = E \Rightarrow \lim_{V \rightarrow \infty} \mu_\beta^V(O) = \lim_{V \rightarrow \infty} \tilde{\mu}_E^V(O)$$

and μ 's are “**equivalent** in the thermodynamic limit”.

Ruelle's generalization of EH unifies equilibrium and nonequilibrium and assumes that generically chaotic systems are such in the precise sense of Axiom A implies also in nonequilibrium \Rightarrow unique statistics for the stationary states.

Axiom A can be simplified (Cohen & G) replacing it with “generically” chaotic motions evolve towards a smooth attracting surface over which motion is chaotic in the sense of Anosov (stronger than Axiom A): Chaotic hypothesis or CH.

Hence is natural to ask whether a theory of ensembles (*i.e.* of families of distr.) in 1-to-1 correspondence yield equivalent statistical descriptions for same stationary state.

Any theory of large (macrosc.) systems \Rightarrow requires

- (1) Regularization of equations (via a "cut off")
- (2) Restriction on observables ("local observables")

Regularization, necessary in essentially all cases, replaces $\dot{\mathbf{u}} = f_R(\mathbf{u})$ (∞ -dim) by a regularized $\dot{\mathbf{u}} = f_R^V(\mathbf{u})$ ($< \infty$ -dim).

Stationary $\mu_R^V(d\mathbf{u})$ uniquely determined by Ruelle's extension of ergodic hypothesis (*i.e.* SRB distrib.).

Form a family \mathcal{E}_R^V of distributions assigning average values to the restricted observables.

For instance:

(a) In Stat. Mech: **local observables** and cut-off $V =$ **container size**; \Rightarrow find their averages **at limit** as $V \rightarrow \infty$:

(b) In Fluid Mech.: **large scale observables** (*i.e.* functions of velocities with “waves” $|\mathbf{k}| < K \ll N$) and cut-off N on the maximum wave $|\mathbf{k}|$: \Rightarrow find averages **at limit** as $N \rightarrow \infty$

Concentrate on the paradigmatic case of periodic NS fluid, [1, 2].

(a) 2/3-Dim., **incompressible**,

(b) **fixed large scale forcing** F (e.g. **with only one or few** Fourier's waves and $\|F\|_2 = 1$),

(c) dissipate heat via **viscosity** $\nu = \frac{1}{R}$

$$NS_{irr}: \dot{u}_\alpha = -(\mathbf{u} \cdot \boldsymbol{\partial})u_\alpha - \partial_\alpha p + \frac{1}{R}\Delta \mathbf{u}_\alpha + F_\alpha, \quad \partial_\alpha u_\alpha = 0$$

$$\text{Velocity: } \mathbf{u}(x) = \sum_{\mathbf{k} \neq \mathbf{0}} u_{\mathbf{k}} \frac{i\mathbf{k}^\perp}{|\mathbf{k}|} e^{i\mathbf{k} \cdot \mathbf{x}}, \quad \bar{u}_{\mathbf{k}} = u_{-\mathbf{k}} \quad (\text{NS-2D})$$

$$NS_{2,irr}: \dot{u}_{\mathbf{k}} = - \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} \frac{(\mathbf{k}_1^\perp \cdot \mathbf{k}_2)(\mathbf{k}_2^2 - \mathbf{k}_1^2)}{2|\mathbf{k}_1||\mathbf{k}_2||\mathbf{k}|} u_{\mathbf{k}_1} u_{\mathbf{k}_2} - \nu \mathbf{k}^2 u_{\mathbf{k}} + f_{\mathbf{k}}$$

“Regularize eq.”: waves $|\mathbf{k}_j| \leq N$. At UV -Cut-off, N .

Remark: $Iu_\alpha = -u_\alpha$ implies solutions $t \rightarrow S_t^{irr} \mathbf{u}$ s.t.

$$IS_t^{irr} \neq S_{-t}^{irr} I \quad \Rightarrow \quad \text{irreversibility}$$

Given init. data u , evolution $u \rightarrow S_t^{irr} u$ generates a steady state (*i.e.* a SRB probability distr.) $\mu_R^{irr,N}$ on M_N .

Unique out a 0-volume of u 's, for simplicity AT R small: “NS gauge symmetry” exists.; phase transitions, [3, 4, 5].

As R varies steady distr. $\mu_R^{irr,N}(du)$ are collected in $\mathcal{E}^{irr,N}$: This is A statistical ensemble of stationary nonequilibrium distrib. for NS_{irr} .

Average energy E_R , average dissipation En_R , Lyapunov spectra (local and global) ... will be defined, *e.g.*:

$$E_R = \int_{M_N} \mu_R^{irr,N}(du) \|u\|_2^2, \quad En_R = \int_{M_N} \mu_R^{irr,N}(du) \|\mathbf{k}u\|_2^2$$

Are there other ensembles whose elements can be put in 1 – 1 correspondence with the ones in $\mathcal{E}^{irr,N}$ and give same average values to “local observables”?

Consider **new equation**, NS_{rev} (with cut-off N):

$$\dot{\mathbf{u}}_{\mathbf{k}} = \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} \frac{(\mathbf{k}_1^\perp \cdot \mathbf{k}_2)(\mathbf{k}_2^2 - \mathbf{k}_1^2)}{2|\mathbf{k}_1||\mathbf{k}_2||\mathbf{k}|} \mathbf{u}_{\mathbf{k}_1} \mathbf{u}_{\mathbf{k}_2} - \alpha(\mathbf{u}) \mathbf{k}^2 \mathbf{u}_{\mathbf{k}} + f_{\mathbf{k}}$$

with α **s. t.** $\mathcal{D}(u) = \|\mathbf{k}u\|_2^2 = En$ (the **enstrophy**) is **exact const of motion** on $u \rightarrow S_t^{rev}u$:

$$\Rightarrow \alpha(u) = \frac{\sum_{\mathbf{k}} \mathbf{k}^2 F_{-\mathbf{k}} u_{\mathbf{k}}}{\sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2} \quad e.g. \quad D = 2$$

New eq. is reversible: $IS_t^{rev}u = S_{-t}^{rev}Iu$ (as α is odd).

α is “**a reversible viscosity**”; (if $D = 3$ α is \sim different)

Rev. eq. is an empirical model of “**thermostat**” on the fluid and **should (?) have same effect of empirical constant friction** (that can also be a thermostat model).

NS_{rev} generates a family of steady states $\mathcal{E}^{rev,N}$ on M_N :
 $\mu_{En}^{rev,N}$ parameterized by constant value of **enstrophy** En .

$\alpha(u)$ in NS_{rev} **will wildly fluctuate** at large R (*i.e.* small viscosity ν) thus “**self averaging**” to a const. value ν
“**homogenizing**” the eq. into NS_{irr} with viscosity ν .

Equivalence mechanism by analogy with Stat. Mech.

- (1) analog of “**local observables**”: functions $O(u)$ which depend only on $u_{\mathbf{k}}$ with $|\mathbf{k}| < K$. “**Locality in momentum**”
- (2) analog of “**Volume**”: just the cut-off N confining the \mathbf{k}
- (3) analog of “**state parameter**”: **viscosity** $\nu = \frac{1}{R}$ (irrev. case) or **enstrophy** En (rev. case).

Equivalence condition : $\mu_{En}^{rev,N}(\alpha) = \frac{1}{R}$

Equivalence is **conjectured** at $N = \infty$ in analogy with the **thermodynamic limit** $V \rightarrow \infty$, for **all** R .

Averages of large scale observables same as $N \rightarrow \infty$ for

$$\mu_R^{irr,N} \in \mathcal{E}^{irr,N} \text{ and } \mu_{En}^{rev,N} \in \mathcal{E}^{rev,N}$$

provided, $\mathcal{D}(\mathbf{u}) \stackrel{def}{=} \sum_{\mathbf{k}} \mathbf{k}^2 |\mathbf{u}_{\mathbf{k}}|^2$ is s.t.

$$\mu_R^{irr,N}(\mathcal{D}) = En, \quad \text{or} \quad \mu_{En}^{rev,N}(\alpha) = \frac{1}{R} = \nu$$

Balance: multiplying NS eq. by $\bar{u}_{\mathbf{k}}$ and sum on \mathbf{k} :

$$\frac{1}{2} \frac{d}{dt} \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 = -\gamma \mathcal{D}(\mathbf{u}) + W(\mathbf{u}), \quad \gamma = \nu \text{ or } \alpha(\mathbf{u})$$

(transport terms = 0, $D = 2, 3$), $\mathcal{D}(\mathbf{u}) = \sum_{\mathbf{k}} \mathbf{k}^2 |u_{\mathbf{k}}|^2 =$
enstrophy and $W = \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}} u_{-\mathbf{k}} =$ **power** of external force.

Remark that W is a local observable

Hence time averaging

$$\frac{1}{R}\mu_R^{irr,N}(\mathcal{D}) = \mu_R^{irr,N}(W), \quad \mu_{En}^{rev,N}(\alpha)En = \mu_{En}^{rev,N}(W)$$

But W is **local** (as \mathbf{f} is such) and, if the conjecture holds, has equal average under the **equivalence** condition: hence $\mu_R^{irr,N}(\mathcal{D}) = En$ **implies** the relation

$$\lim_{N \rightarrow \infty} R\mu_{En}^{rev,N}(\alpha) = 1$$

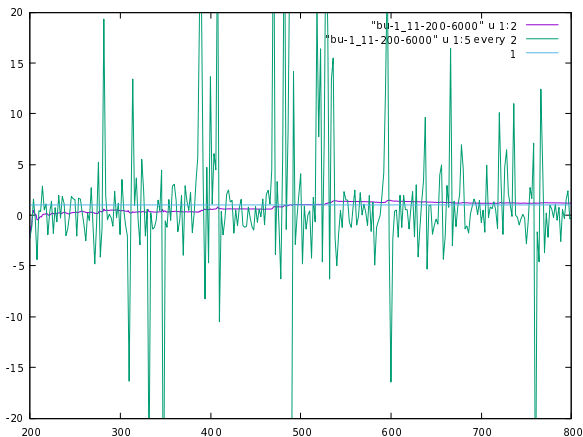
This becomes a **first rather stringent test** of the conjecture.

But it will be useful to pause to illustrate a few preliminary simulations and checks.

Unfortunately the following simulations are in dimension 2 ($D = 3$ is at the moment beyond the available (to me) computational tools) although present day available NS codes should be perfectly capable to perform detailed checks in rapid time, [6].

Concentrate on the first test:

$$\lim_{N \rightarrow \infty} R\mu_{En}^{rev}(\alpha) = 1$$

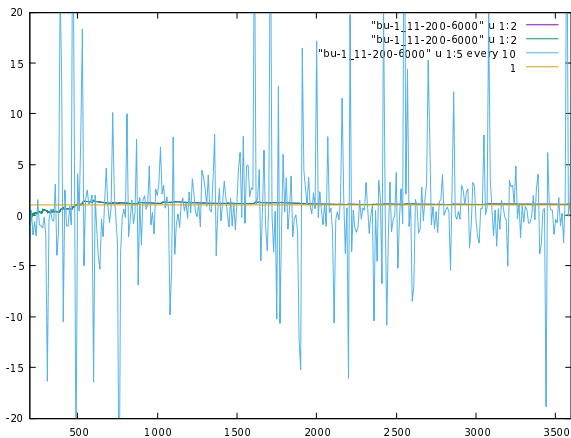


FigA32-19-17-11.1-detail

Fig.1 (detail): Running average of reversible friction

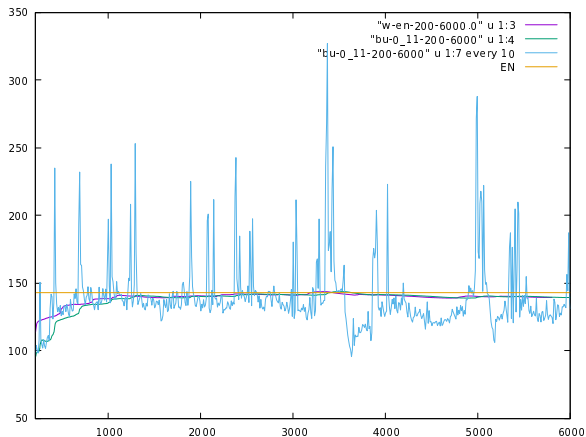
$R\alpha(u) \equiv R \frac{2\text{Re}(f_{-\mathbf{k}_0} u_{\mathbf{k}_0}) \mathbf{k}_0^2}{\sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2}$, superposed to conjectured 1 and to the fluctuating values of $R\alpha(u)$. **Initial transient** $t < 800$.

Evol.: NS_{rev} , **R=2048**, 224 modes, Lyap. $\simeq 2$, x-unit = 2^{19}



FigA32-19-17-11.1-all

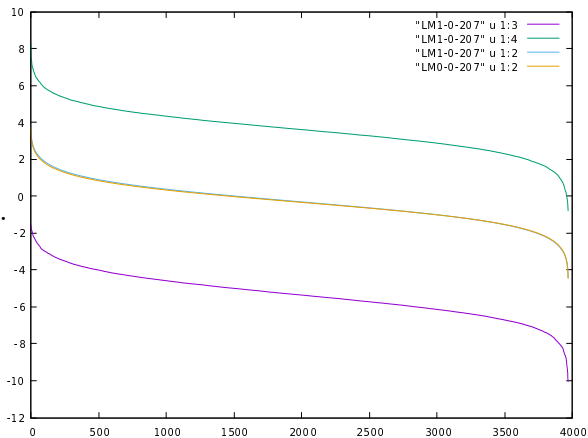
Fig.1-bis: As previous fig. but **time 8 times** longer: data reported “every 10”, **or** black.



FigEN32-19-17-11.1

Fig.2: NS_{irr} : **Running** average of the work $R \sum_{\mathbf{k}} F_{-\mathbf{k}} u_{\mathbf{k}}$ (**violet**) in NS_{rev} ; and **convergence** to average enstrophy En (**orange** straight line), **blue** is running average of enstrophy $\sum_{\mathbf{k}} \mathbf{k}^2 |u_{\mathbf{k}}|^2$ in NS_{irr} , enstrophy **fluctuations** violet in NS_{irr} : **R=2048**.

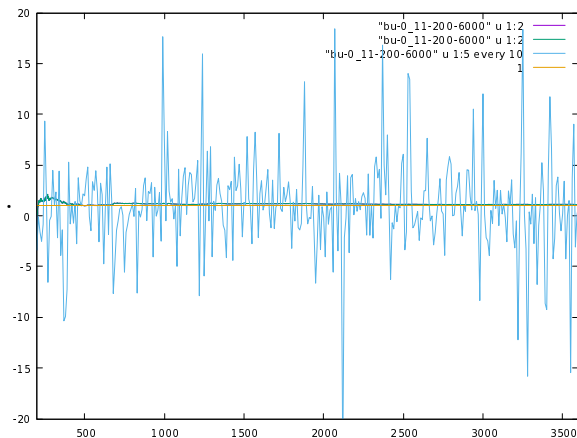
Unexpected



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Fig.3: **Fluct. max.-min.** showing the $N = \mathbf{3968}$ NS_{rev} -exponents. Central lines rev-irr. **superposed** averaged over 800 samples with remarkable coincidence of NS_{rev} with the NS_{irr} exponents. Strong fluct. between max-min variations (upper and lower lines)

The following Fig.4 (similar to Fig.1 but w. NS_{irr}):



FigA32-19-17-11.0-all

Fig.4: As Fig.1 but running average of reversible friction $R\alpha(\mathbf{u})$ regarded as observ. in NS_{irr} , superposed to value 1 and to fluctuating values of $R\alpha(\mathbf{u})$. An extension ? of conjecture since $\alpha(\mathbf{u})$ is not local.

The figure suggests (from the theory of Anosov systems):

Check the “Fluctuation Relation” in the **reversible** evolutions: for the divergence (trace of the Jacobian) $\sigma(u) = -\sum_{\mathbf{k}} \partial_{u_{\mathbf{k}}} (\dot{u}_{\mathbf{k}})_{rev}$: let p (time τ average of $\frac{\sigma}{\langle \sigma \rangle}$)

$$p \stackrel{def}{=} \frac{1}{\tau} \int_0^\tau \frac{\sigma(\mathbf{u}(t))}{\langle \sigma \rangle_{irr}} dt,$$

then a theorem for Anosov systems:

$$\frac{P_{srb}(p)}{P_{srb}(-p)} = e^{\tau \mathbf{1}_P \langle \sigma \rangle_{irr}} \text{ (sense of large deviat. as } \tau \rightarrow \infty \text{)}$$

Replacing reversible evolution with irreversible but studying the **reversible divergence** becomes a “*reversibility test on the irreversible flow*”!

Can this be applied to turbulence ? **However:**

Problem 1: if attracting set \mathcal{A} has lower dimension, time reversal symmetry I **cannot be applied** because $I\mathcal{A} \neq \mathcal{A}$. This **certainly occurs** if N becomes large enough, [7, 8].

Help could come **if** exists further symmetry P between \mathcal{A} and $I\mathcal{A}$ *commuting* with S_t : $PS_t = S_tP$.

Then $P \circ I : \mathcal{A} \rightarrow \mathcal{A}$ **becomes a time reversal symmetry of the motion restricted to \mathcal{A} .**

Problem 2: **even supposing existence** of P , still **is is not** possible to apply FR because, at best, it would concern the contraction $\sigma_{\mathcal{A}}(\mathbf{u})$ of \mathcal{A} and not the $\sigma(\mathbf{u})$ of M_V .

The $\sigma(\mathbf{u})$ receives contributions from the exponential approach to \mathcal{A} : which **obviously do not contribute to $\sigma_{\mathcal{A}}$.**

How to recognize such contributions ?

Help for both problems could come from “pairing property”
Often Lyapunov exponents (local and global) **arise in pairs** with **almost constant average** or average on a regular curve.

In a few systems pairs have an **exactly constant average**.

An idea can be obtained **from the local exponents**
(eigenvalues of the symmetric part of Jacobian matrix).

For instance NS seems to enjoy a pairing rule: the following figure illustrates it (at strong regularization and the large Reynolds number); looks close to exact (**at graph scale!**).

Pairing in NS **cannot** stay at all regularizations: the above should be considered due to the small N . However it has been proposed that the apparent “**pairing line**” becomes at large N a “**pairing curve**” (as apparently so at small R).

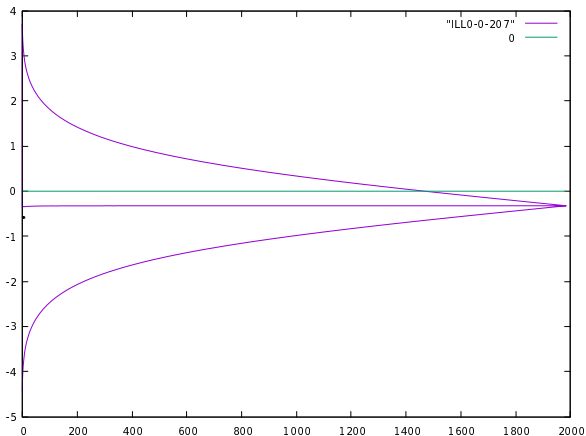


FIG11-128-15-14-11

Fig.5: $R = 2048$, **3968modes**, **local** exponents ordered decreasing: s.t. λ_k , $0 \leq k < d/2$, and increasing λ_{d-k} , $0 \leq k < d/2$, the line $\frac{1}{2}(\lambda_k + \lambda_{d-1-k})$ and the line $\equiv 0$. **Irreversible case** and **apparent pairing rule** and **dimensional loss** $\varphi \simeq \frac{1500}{1980}$.

In Anosov reversible systems the number of positive exp. = number of negative: hence **proposal**, [9, p.445],[10],

“attracting surface \mathcal{A} dimension = **twice the number of positive exponents**” and

if pairing: **twice** num. of opposite sign pairs.

Implication: $\sigma_{\mathcal{A}}(\mathbf{u})$ is proportional to the total $\sigma(\mathbf{u})$ if pairing to a constant

$$\sigma_{\mathcal{A}}(\mathbf{u}) = \varphi \sigma(\mathbf{u}), \quad \varphi = \frac{\text{number of opposite pairs}}{\text{total number of pairs}}$$

(for pairing to a more general curve

$$\sigma_{\mathcal{A}}(u) = \sigma(u) + \sum_{\text{pairs} < 0} (\lambda_j + \lambda'_j)). \quad \text{Why?}$$

Idea: negative pairs correspond to exponents associated with attraction to \mathcal{A} : **hence irrelevant in computing $\sigma_{\mathcal{A}}$.**

Assuming **existence of P** , hence time-reversal on attractor, and **pairing + CH \Rightarrow FR but slope $\varphi < 1$** :

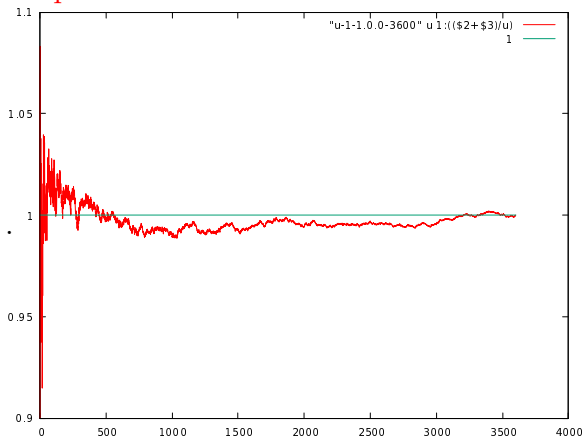
$$\tau p \varphi \sigma, \quad \text{rather than } \tau p \sigma : \quad \text{in fig. } \varphi \simeq \frac{1550}{2490}$$

If true: this will be a “check of reversibility” in NS_{irr} .

More elaborate checks are being attempted: [6, 11] +

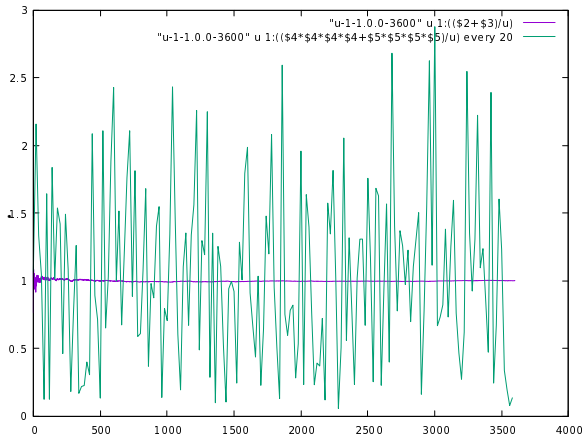
- (a) **moments** of large scale observables rev & irr
- (b) local Lyap. exponents of **matrices** different from Jacobian
- (c) check of the **fluctuation rel.**, particularly in irrev. cases, (shown above to be accessible already with **960 modes** and $R = 2048$): \Rightarrow **FR with slope $\varphi < 1$** (Axiom C ?), [9, 12].
- (d) More values of R and N an example **with R larger than in the preceding cases yields similar results (not shown)**.

Example of moments of local observables:



FIGu0-64-191711-10

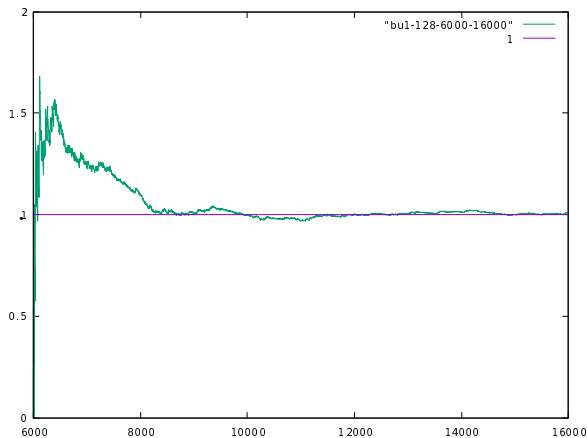
Fig.6: Running averages in NS_{rev} of $(|Re u_{11}|^4 + |Im u_{11}|^4) / \langle |Re u_{11}|^4 + |Im u_{11}|^4 \rangle_{irr}$,
 $R = 2048, 960$ modes. Conjecture yields ratio tending to 1



FIGu1-64-191711-10

Fig.7: Same running averages in NS_{rev} of $(|Re u_{11}|^4 + |Im u_{11}|^4) / \langle |Re u_{11}|^4 + |Im u_{11}|^4 \rangle_{irr}$, and their rev. fluctuations, for $R = 2048, 960$ modes.

Concluding the simulation



FIGA-128

Fig.8: Illustration of the conjecture on a **3968 modes** NS: the running averages of $R\alpha$ in the **reversible** NS should tend to 1, **according to conjecture**.

Finally rigorous estimate of number \mathcal{N} of Lyap. exp. needed so that their sum remains > 0 :

$$\lesssim \sqrt{2}A(2\pi)^2\sqrt{R}\sqrt{R}En, A = 0.55..$$

in dimension 2, while at dimension 3 a similar estimate holds but it involves a norm different from the enstrophy. (Ruelle if $d = 3$ and Lieb if $d = 2, 3$, [13, 8]).

Applied here it would require $\mathcal{N} \sim 2.10^4$ for NS 2D: **not accessible** in the simulations presented here but **not impossible** in principle with available computers and computation methods already available, at least if $D = 2$.

Finally **further** careful checks are required, particularly since inspiring ideas are, **to say the least**, **controversial** as shown by quotes from a well known treatise, [14, p.344-347] and [2, app.A].

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