Kondo problem: paradigm of RG application

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1-d lattice, fermions+impurity, "Kondo problem"

$$H_h = \sum_{x=-L/2}^{L/2-1} \psi^+(x)\,(-\frac{1}{2}\Delta - 1)\,\psi^-(x) + h\ \tau^z$$

$$H_K = H_h - \lambda\psi^+(0)\sigma^j\psi^-(0)\,\tau^j \;=\; H_h + V$$

(1) $\psi^\pm_\alpha(x)$ C&A operators, σ^j, τ^j, $j = 1, 2, 3$, Pauli matrices

(2) $x \in$ unit lattice, $-L/2$, $L/2$ identified (periodic b.c.)

(3) $\Delta f(x) = f(x + 1) - 2f(x) + f(x - 1)$ discrete Laplacian.

Alternative model $\boldsymbol{\tau} \to d^+\boldsymbol{\tau}d^-$ with $d^\pm$ fermion (Wilson, Andrei models).

$\lambda < 0$=antiferrom., $\lambda > 0$ ferrom.

No interaction ($\lambda = 0$): 1 impurity and $\beta h < 1$ (*e.g.* $h = 0$)

$$\chi(\beta, h) \propto \beta \xrightarrow[\beta\to\infty]{} \infty, \qquad \forall\ L \geq 1,\ \beta h < 1$$

Interaction (classical) 1 elec.&1 impurity:
1) field on impurity & $\lambda \neq 0$

$$\chi(\beta, 0) = 0 \quad \text{repulsive}, \qquad +\infty \quad \text{attractive}$$

2) Still true if $L < \infty$ classic&quantum or $L = \infty$ classic

Reason **????**: $\lambda < 0$ $\rightarrow$ rigidly antiparallel spins

Then Trivial? (0 repulsive, ∞ attractive ?)

BUT

If $L = \infty$ quantum chain: new phenomena

1) no impurity: $\Rightarrow$ Pauli paramagnetism (1926)

local (or specific) magnetic suscept. $< \infty$ at $T \geq 0$:

$$\chi(\infty, 0) = \rho \frac{1}{k_B T_F} \frac{d}{2}, \qquad (Pauli)$$

2) at fixed $\lambda < 0 \Rightarrow$ Kondo effect:

susceptibility $\chi(\beta, h)$ (unlike XY: spin essential)

smooth and > 0 at $T = 0$ and $h \geq 0$

Kondo realized the problem (3^d-order P.T.) and gave arguments (1964) for $\chi < \infty$ (actually conductivity $< \infty$)

Anderson-Yuval-Hamann (1969,70) $\Rightarrow$ multiscale nature, relation with 1D Coulomb gas & (no Kondo eff. $\lambda < 0$), &

& stress lack of asymptotic freedom = obstacle for $\lambda > 0$.
Later Andrei (1980) provided an exact solution of a closely related model.
Earlier Wilson (1974-1975) had overcome lack of asympt. freedom: simplified model and a recursion scheme, $\frac{1}{2}$-numerically.

Method builds sequence of approximate Hamiltonians more and more accurately representing the system on larger and larger scales, with Kondo effect via a nontrivial fixed point.

Evaluate $Z = \mathrm{Tr}\, e^{-\beta H_K}$ via Wick's rule.

$$Z = \mathrm{Tr}\left\langle \sum_{n=0}^{\infty} (-1)^n \int_{0<t_1<\cdots<t_n<\beta} dt_1 \cdots dt_n\, V(t_1) \cdots V(t_n) \right\rangle$$

$$V(t) \stackrel{def}{=} -\lambda_0 \psi^+(t) \sigma^j \psi^-_{\alpha_2}(t)\, \tau^j - h\, \boldsymbol{\omega}_j \tau^j$$

Averages of observables depending only on the site 0 (*e.g.* impurity susceptibility) require by Wick $\Rightarrow$ only Feynman graphs with propagators at $x = 0$: $g(t - t')$:

$$g(t - t') = \sum_{\omega=\pm} \int \frac{dk_0 dk}{(2\pi)^2} \frac{e^{ik_0(t-t')}}{-ik_0 + \omega k} \chi(k_0^2 + k^2),$$

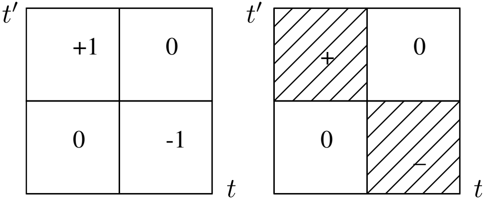
here a first simplification: cut-off of the large k, k_0 and linear dispersion relation $\pm k$ at the Fermi level $k = 0$).
The multiscale decomposition of g

$$g(t - t') = \sum_{m=0}^{-\infty} 2^m g_0(2^m(t - t'))$$

exhibits the scaling properties of g: namely the long range $\sim \frac{1}{t-t'}$ decomposed as a sum of short range propagators identical up to scaling.

The hierarchical model introduces a further simplification

$$
\begin{aligned}
g(t-t') &= \sum_{m=0}^{-\infty} 2^m g_0(2^m(t-t')) \\
g_0(t,t') &= 0 \text{ unless } t,t' \in [k,k+1] \\
g_0(t,t') &= \begin{cases} 1 & \text{if } t \in [k,k+\frac{1}{2}] \text{ and } t' \in [k+\frac{1}{2},k] \\ -1 & \text{if } t' \in [k,k+\frac{1}{2}] \text{ and } t \in [k+\frac{1}{2},k] \end{cases} \\
g_0(t,t') &= 0 \qquad \text{otherwise}
\end{aligned}
$$



g_0 looses translation invariance but the propagator g keeps the multiscale and long range properties of the initial model, at least hierarchically

But since the impurity is localized observ. localized at 0 depend on fields at 0, $\psi^{\pm}(0), \varphi^{\pm}$ $\Rightarrow$ **1D problem** (AYH).

Illustration of (AYH970) remark: 1D problem, (long range) Main operators in the Lagrangian:

$$O_0(t) \stackrel{def}{=} \psi^+(t)\boldsymbol{\sigma}\psi^-(t)\cdot\boldsymbol{\tau} = \vec{A}(t)\cdot\boldsymbol{\tau}, \qquad O_5(t) \stackrel{def}{=} \boldsymbol{\tau}\cdot\boldsymbol{\omega}$$

(in Grassmannian form) and

$\mathcal{L}_K$ on scale m is (with $\alpha_0 < 0, \alpha_5 = h \geq 0$ else 0).

$$\int e^{\mathcal{L}_K^{[<=m]}(\psi^{[\le m]})} d\psi = \int e^{-\int_0^\beta \sum_i \alpha_i^{[m]} O_i(t)\,dt} d\psi^{[0]} d\psi^{[1]} \ldots d\psi^{[m+1]}$$

Set RG analysis via (Grassmannian) for $\mathrm{Tr}\, e^{-\beta H_K}$

Key: IF $h = 0$ then $\mathcal{L}_K^{[m]}(t)$ is $\forall m$:

$$\alpha_0^{[m]} O_0(t)\cdot\boldsymbol{\tau} + \alpha_1^{[m]} O_1(t)$$

i.e. no new operators needed at any scale (exact recursion)

The fermionic field $\psi^{\pm}(x,t)$ is then represented as

$$\psi^{\pm}(x,t) = \sum_{m=0}^{-\infty} 2^{\frac{m}{2}} \psi^{[m],\pm}(x,t)$$

Three relations used to compute the flow equation, which follow from a patient algebraic meditation:

$$\langle\, A_1^{j_1} A_2^{j_2} \,\rangle = \delta_{j_1,j_2}\left(2 + \frac{1}{3}\vec{a}^2\right) - 2\, a^{j_1,j_2}\delta_{j_1\neq j_2}\, s_{t_2,t_1}$$
$$\langle\, A_1^{j_1} A_1^{j_2} A_2^{j_3} \,\rangle =\equiv 2\, a^{j_3}\, \delta_{j_1,j_2}$$
$$\langle\, A_1^{j_1} A_1^{j_2} A_2^{j_3} A_2^{j_4} \,\rangle = 4\delta_{j_1,j_2}\delta_{j_3,j_4}$$

where the lower case $\vec{a}$ denote $\langle\, \vec{A}_1 \,\rangle \equiv \langle\, \vec{A}_2 \,\rangle$ and $a^{j_1,j_2} = \langle\, \psi_1^+ \sigma^{j_1} \sigma^{j_2} \psi_1^- \,\rangle = \langle\, \psi_2^+ \sigma^{j_1} \sigma^{j_2} \psi_2^- \,\rangle$.

Remark the model calculations only involve fields localized at the impurity site, $x = 0$: $\Rightarrow$ we deal with 1-dimensional fermionic fields.

This does not mean that the lattice supporting the electrons plays no role: on the contrary it shows up, and in an essential way, because the "dimension" of the electron field will be different from that of the impurity, as evident from the factor $2^{\frac{m}{2}} \xrightarrow[m\to-\infty]{} 0$.

$$O_{0,\eta}^{[\le 0]}(\Delta) \stackrel{def}{=} \frac{1}{2}\mathbf{A}_\eta^{[\le 0]}(\Delta)\cdot\boldsymbol{\tau}, \quad O_{4,\eta}^{[\le 0]}(\Delta) \stackrel{def}{=} \frac{1}{2}\mathbf{A}_\eta^{[\le 0]}(\Delta)\cdot\boldsymbol{\omega},$$

$$O_{1,\eta}^{[\le 0]}(\Delta) \stackrel{def}{=} \frac{1}{2}\mathbf{A}_\eta^{[\le 0]}(\Delta)^2, \quad O_{6,\eta}^{[\le 0]}(\Delta) \stackrel{def}{=} \frac{1}{2}(\mathbf{A}_\eta^{[\le 0]}(\Delta)\cdot\boldsymbol{\omega})(\boldsymbol{\tau}\cdot\boldsymbol{\omega})$$

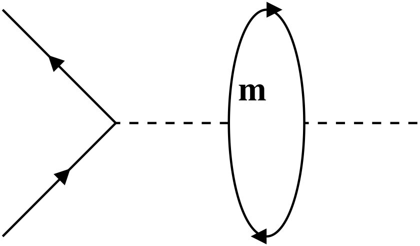
$$O_{5,\eta}^{[\le 0]}(\Delta) \stackrel{def}{=} \frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{\omega}, \quad O_{7,\eta}^{[\le 0]}(\Delta) \stackrel{def}{=} \frac{1}{2}(\mathbf{A}_\eta^{[\le 0]}(\Delta)^2)(\boldsymbol{\tau}\cdot\boldsymbol{\omega})$$

Scaling O_0 = marginal, O_1 irrrelevant,O_5 = relevant

The RG consists in

1) Expand perturbatively $Z^{[>m]} = e^{V^{[m]}}$ via Feynman gr. heavily using the hierarchical structure

2) Decompose propagators as $\sum_{m=0}^{-\infty} 2^m g_0(2^m(t-t')$



3) Recognize: at $h \neq 0$ no new operators can arise besides

$$O_4 = \vec{A} \cdot \vec{h},\ O_5 = \boldsymbol{\tau} \cdot \vec{h},\ O_6 = \vec{A} \cdot \vec{h}\, \boldsymbol{\tau} \cdot \vec{h}, O_7 = \vec{A}^2 \boldsymbol{\tau} \cdot \vec{h},$$

3) Recognize that the result contains a few series that can collected to form a sequence of running couplings

$$\boldsymbol{\alpha}^{[m]} = (\alpha_0^{[m]}, \alpha_1^{[m]}, \alpha_4^{[m]}, \alpha_5^{[m]}, \alpha_6^{[m]}, \alpha_7^{[m]}).$$

with only $\alpha_0^{[m]}, \alpha_1^{[m]} \neq 0$ if $h = 0$

4) Each is a convergent series in the initial couplings α_0, h, if small enough (BUT converg. radius m dependent)

5) Recognize that the $\boldsymbol{\alpha}^{[m]}$ satisfy a formal recursion

$$\boldsymbol{\alpha}^{[m]} = \Lambda \boldsymbol{\alpha}^{[m+1]} + \mathcal{B}(\boldsymbol{\alpha}^{[m+1]})$$

and $\mathcal{B}$ can be expressed as a "polynomial" with coefficients which are geometric series in $\boldsymbol{\alpha}^{[m+1]}$; $\Lambda = (1, \frac{1}{2}, 1, 2, 1, \frac{1}{2})$.

Even forgetting convergence, PT of no use: marginal term grows (if $\lambda_0 < 0$) and generates growing ("relevant" terms)!

6) Sum the geometric series to obtain a closed from of $\mathcal{B}$.
After a natural change of variables $\boldsymbol{\alpha} \longleftrightarrow \boldsymbol{\lambda}$ at $h = 0$

$$\lambda'_0 = \frac{1}{C}(\lambda_0 + 3\lambda_0\lambda_1 - \lambda_0^2)$$

$$\lambda'_1 = \frac{1}{C}(\frac{1}{2}\lambda_1 + \frac{1}{8}\lambda_0^2),$$

$$C = 1 + \frac{3}{2}\lambda_0^2 + 9\lambda_1^2$$

Non perturbative: for $m \to -\infty$ (IR limit, $\beta = +\infty$, $T = 0$)

$\boldsymbol{\lambda}^{[m]}, \boldsymbol{\alpha}^{[m]}$ converge to non trivial fixed point

if $h = 0, \alpha_0 < 0$, exactly computable,
$\lambda_0^* = -7.807257...10^{-1}, \lambda_1^* = 5.292875...10^{-2}$

$$\lambda_0^* = -x\frac{1+5x}{1-4x},\ \lambda_1^* = \frac{x}{3},\ x = 7.807257...10^{-1},$$

with $4 - 19x - 22x^2 - 107x^3 = 0$, real root.

Susceptibility: new operators needed to close beta

$$O_4 = \vec{A} \cdot \vec{h},\ O_5 = \boldsymbol{\tau} \cdot \vec{h},\ O_6 = \vec{A} \cdot \vec{h}\, \boldsymbol{\tau} \cdot \vec{h}, O_7 = \vec{A}^2 \boldsymbol{\tau} \cdot \vec{h},$$

O_0, O_4, O_6 marginal, O_5 relevant, O_1, O_7 irrelevant

Calculating beta function: via Feynman graphs, after simplifications, a beta function with 36 coeff is found

From the flow of the $\boldsymbol{\alpha}$ the partition function $Z(\beta, h)$ is computed and susceptibility

$$\chi(\beta, h) = \partial_h^2 \log Z(\beta, h)$$

follows as a function of h.

The beta function is a rational function defined by the ratio of two polynomials of degree 2.

$$C = 1 + \lambda_0^2 + \frac{1}{2}(\lambda_0 + \lambda_6)^2 + 9\lambda_1^2 + \frac{1}{2}\lambda_4^2 + \frac{1}{4}\lambda_5^2 + 9\lambda_7^2$$

$$\lambda_0' = \frac{1}{C}(\lambda_0 - \lambda_0^2 + 3\lambda_0\lambda_1 - \lambda_0\lambda_6)$$

$$\lambda_1' = \frac{1}{C}(\frac{1}{2}\lambda_1 + \frac{1}{8}\lambda_0^2 + \frac{1}{12}\lambda_0\lambda_6 + \frac{1}{24}\lambda_4^2 + \frac{1}{4}\lambda_5\lambda_7 + \frac{1}{24}\lambda_6^2)$$

$$\lambda_4' = \frac{1}{C}(\lambda_4 + \frac{1}{2}\lambda_0\lambda_5 + 3\lambda_0\lambda_7 + 3\lambda_1\lambda_4 + \frac{1}{2}\lambda_5\lambda_6 + 3\lambda_6\lambda_7)$$

$$\lambda_5' = \frac{1}{C}(2\lambda_5 + 2\lambda_0\lambda_4 + 36\lambda_1\lambda_7 + 2\lambda_4\lambda_6)$$

$$\lambda_6' = \frac{1}{C}(\lambda_6 + \lambda_0\lambda_6 + 3\lambda_1\lambda_6 + \frac{1}{2}\lambda_4\lambda_5 + 3\lambda_4\lambda_7)$$

$$\lambda_7' = \frac{1}{C}(\frac{1}{2}\lambda_7 + \frac{1}{12}\lambda_0\lambda_4 + \frac{1}{4}\lambda_1\lambda_5 + \frac{1}{12}\lambda_4\lambda_6)$$

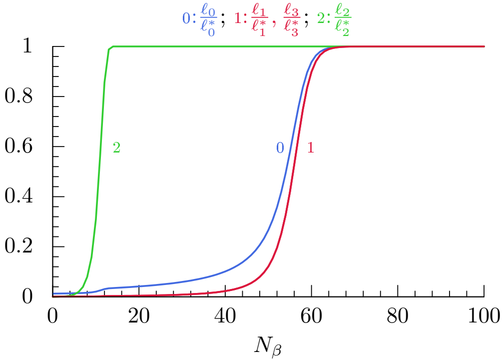


Fig.2: plot of $\frac{\lambda_i}{\lambda_i^*}$, $i = 0, 1$, as a function of $N_\beta = \log_2 \beta$,
$\lambda_0 \equiv \alpha_0 = -0.1, -0.01$ respectively the left and the right pairs.

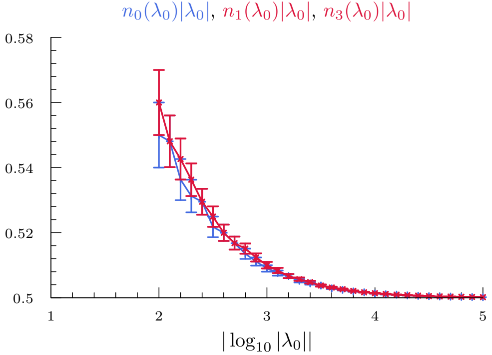


Fig.3: inflection point $n_0(\lambda_0)$: $n_0(\lambda_0)\lambda_0$ vs. $|\log_2|\lambda_0||$: only data with 10% error (upper and lower curves) visual lines interpolate data. The T_K for λ_0 small tends to a constant.

$$T_K = const\, e^{-c_0\lambda_0^{-1}}$$

For $h \neq 0$ the flow leads to “high T fixed pt.” at scale $n(h)h \xrightarrow[h\to 0]{} c$, $i.e.\ T_h = e^{-ch^{-1}}$

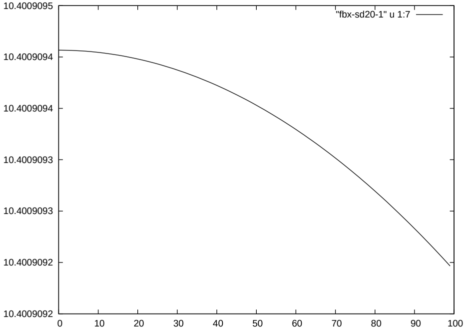


Fig.4: plot of $\chi(\beta, h)$ for $h \in [0, 10^{-6}]$ at $\lambda_0 = -0.3$ and $\beta = 2^{20}$ (so that the largest value for βh is ~ 1)
[1, 2, 4, 3, 5]

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$$\ell_2^{[m]} = \frac{1}{3}, \quad \ell_3^{[m]} = \frac{1}{6}\ell_1^{[m]}, \quad \ell_8^{[m]} = \frac{1}{6}\ell_4^{[m]}.$$