Nonequilibrium and statistical ensembles

Statistical properties of an Equilibrium state are obtained by several different probability distributions, *e.g.* canonical or microcanonical: which attribute the same average to physically interesting obervables. **Reminder**:

The probability distr. describing a system with ρV particles in volume V can be collected in families $\mathcal{E}^{mc}, \mathcal{E}^{c}, \ldots$ whose elements are parameterized by E, β,...

1) observables of interest are local observables $O \in \mathcal{O}_{loc}$: $O(\mathbf{p}, \mathbf{q})$ depending on $\mathbf{p}, \mathbf{q}$ only through coordinates of particles $q_i \in \mathbf{q}$ with $q_i \in \Lambda$ where Λ is a volume $\ll V$

2) the probability distribution $\mu_\beta^V \in \mathcal{E}^c$ and $\widetilde{\mu}_E^V \in \mathcal{E}^{mc}$ are **correspondent** if β, E are s.t.

$$\mu_\beta^V(H_V(\mathbf{p}, \mathbf{q})) = E$$

Then

$$\lim_{V\to\infty} \mu_\beta^V(O) = \lim_{V\to\infty} \widetilde{\mu}_E^V(O)$$

and μ's are *equivalent* in the thermodynamic limit.

In case of phase transitions extra labels $\gamma, \widetilde{\gamma}$ are added to identify the extremal distributions and it is possible to establish a correspondence between the extra labels $\gamma \longleftrightarrow \widetilde{\gamma}$ so that the equivalence can be equally formulated.

Is it possible a similar description of the stationary states of nonequilibrium systems?

Think an evolution eq. of u on a "phase space" M depending on a parameter R:

$$\dot{\mathbf{u}} = \mathbf{f_R}(\mathbf{u})$$

Typically eq. will be difficult and even existence-1-qness will be open problems.

As an example consider infinitely many hard spheres of given density or a forced incompressible 3D NS fluid with periodic b.c.: at best only not constructive"weak solutions".

$\Rightarrow$ the eq. will have to be regularized in $f_R^V(u)$ where V is a regularization parameter.

E.g. **(1)** in **SM** V is typically the container size: and the problem becomes finding the observables whose averages have a limit as $V \to \infty$: Physical observables.

Such observables $u \to O(u)$ in SM are those only depending on points of u in region $K \ll V$, "local observables".

(2) For the NS equation the regularization parameter could be a "UV cut-off" N. And it is natural to consider as physical observables the $u \to O(u)$ which only depend on the Fourier's components $\mathbf{k}$ of u with $|\mathbf{k}| < K \ll N$.

Once the class of physical observables is restricted, it is to be expected (?) that several equations of motion could describe the stationary states of the same system.

E.g. the h.c. system can be described by the Hamilton eq.s but also by the isothermal equations

$$\dot{\mathbf{q}} = \mathbf{p}, \qquad \dot{\mathbf{p}} = -\boldsymbol{\partial}_{\mathbf{q}} V(\mathbf{q}) - \alpha(\mathbf{p},\mathbf{q})\mathbf{p}$$

where $\alpha(\mathbf{p},\mathbf{q})$ is a multiplier which imposes $T(\mathbf{p}) = const.$

Stationary states of the two equations are parameterized by energy E or by kinetic energy T; stationary states will be

$$\mu_E^{mc,V} = \delta(H(\mathbf{p},\mathbf{q}) - E)d\mathbf{p}d\mathbf{q} \qquad or, respectively:$$

$$\mu_\beta^{c,V} = e^{-\beta_0 V(\mathbf{q})}\delta(T(\mathbf{p}) - N\beta^{-1})d\mathbf{p}d\mathbf{q}, \qquad \beta_0 = \beta(1 - \frac{1}{3N})$$

Equivalent if
$\mu_\beta^{c,V}(H) = E \,:\, \lim_{V\to\infty} \mu_\beta^{c,V}(O) = \lim_{V\to\infty} \mu_E^{mc,V}(O).$

Interesting cases arise when the system is described by equations which obey a symmetry but they are phenomenologically described by non symmetric equations (cases of spontaneously broken symmetry).

Consider, as a typical case, the Navier-Stokes equation: in the case of the above incompressible fluid they can be regarded as Euler equations subject to a thermostat absorbing the heat due to the viscosity: which turns the equations into time-reversal breaking ones.

A paradigmatic case is a fluid in a periodic container 2/3-Dim., incompressible, **at fixed forcing** F (smooth, *e.g.* with only one Fourier component non zero and $\|F\|_2 = 1$), kept at const. temp. by a thermostat. to dissipate heat via the force due to viscosity $\boldsymbol{\nu} = \frac{\mathbf{1}}{\mathbf{R}}$ (consistently with incompressibility).

NS_{irr}: $\dot u_\alpha = -(\vec u\cdot\boldsymbol{\partial})u_\alpha - \partial_\alpha p + \frac{1}{R}\Delta u_\alpha + F_\alpha, \qquad \partial_\alpha u_\alpha = 0$

Velocity: $\vec u(x) = \sum_{\vec k\neq\vec 0} u_{\mathbf k}\frac{i\mathbf k^\perp}{|\mathbf k|}e^{i\mathbf k\cdot\mathbf x}, \quad \overline u_{\mathbf k} = u_{-\mathbf k}$ (NS-2D)

$NS_{2,irr}$: $\dot u_{\mathbf k} = \sum_{\mathbf k_1+\mathbf k_2=\mathbf k}\frac{(\mathbf k_1^\perp\cdot\mathbf k_2)(\mathbf k_2^2-\mathbf k_1^2)}{2|\mathbf k_1||\mathbf k_2||\mathbf k|}u_{\mathbf k_1}u_{\mathbf k_2} - \nu\mathbf k^2 u_{\mathbf k} + f_{\mathbf k}$

Imagine to truncate eq. supposing $|\mathbf k_j|\le N$. Cut-off UV, N, is temporarily fixed (**BUT interest is on $N\to\infty$**).

NS 2D $\to$ ODE in a phase space M_N with $4N(N+1)$ dimen. (In **3D** $O(8N^3)$). Exist. & 1-ness trivial $D=2,3$.

BUT $Iu_\alpha = -u_\alpha$ implies $IS_t^{irr}\neq S_{-t}^{irr}I$, $\Rightarrow$: irreversibility.

Given init. data u, evolution $t\to S_t^{irr}u$ generates a steady state (*i.e.* a SRB probability distr.) $\mu_R^{irr,N}$ on M_N.

Unique aside a volume 0 of u's, for simplicity

[Not so at small R: "NS gauge symmetry" exists.**??**]

[1, 2, 3]. As R varies the steady distr. $\mu_R^{irr,N}(du)$ form a collection $\mathcal{E}^{irr,N}$: to be named

A statistical ensemble of stationary nonequilibrium distrib. for NS_{irr}.

And average energy E_R, average dissipation En_R, Lyapunov spectra (local and global) ... will be defined, *e.g.*:

$$E_R = \int_{M_N} \mu_R^{irr,N}(du)||u||_2^2, \qquad En_R = \int_{M_N} \mu_R^{irr,N}(du)||\mathbf{k}u||_2^2$$

Consider new equation, NS_{rev}:

$$\dot{\mathbf{u}}_{\mathbf{k}} = \sum_{\mathbf{k_1}+\mathbf{k_2}=\mathbf{k}} \frac{(\mathbf{k}_1^\perp \cdot \mathbf{k_2})(\mathbf{k}_2^2 - \mathbf{k}_1^2)}{2|\mathbf{k_1}||\mathbf{k_2}||\mathbf{k}|} \mathbf{u}_{\mathbf{k_1}} \mathbf{u}_{\mathbf{k_2}} - \alpha(\mathbf{u})\mathbf{k}^2 \mathbf{u}_{\mathbf{k}} + f_{\mathbf{k}}$$

with α **s. t.** $\mathcal{D}(u) = ||\mathbf{k}u||_2^2 = En$ (the enstrophy)is **exact const of motion** on $u \to S_t^{rev} u$.:

$$\Rightarrow \alpha(u) = \frac{\sum_{\mathbf{k}} \mathbf{k}^2 F_{-\mathbf{k}} u_{\mathbf{k}}}{\sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2} \qquad e.g. \;\; D = 2$$

New eq. is reversible: $IS_t^{rev}u = S_{-t}^{rev}Iu$ (as α is odd).

α is "a reversible viscosity"; (if $D = 3$ α is ∼different)

Rev. eq. can be considered as model of empirical "thermostat" acting on the fluid and should (?) have **same effect** of empirical constant friction.

NS_{rev} generates a family of steady states $\mathcal{E}^{rev,N}$ on M_N: $\mu_{En}^{rev,N}$ parameterized by constant value of **enstrophy** En.

$\alpha(u)$ in NS_{rev} **will wildly fluctuate** at large R (*i.e.* small viscosity ν) thus "**self averaging**" to a const. value ν "homogenizing" the eq. into NS_{irr} with viscosity ν.

Of course we could impose a multiplier $\alpha'(u) = \frac{\sum_{\mathbf{k}} f_{\mathbf{k}}\overline{u}_{\mathbf{k}}}{\sum_{\mathbf{k}} |\mathbf{k}|^2|u_{\mathbf{k}}|^2}$: it would fix energy $E = \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2$ and obtain **diff. rev. eq.**

The equivalence mechanism is suggested by analogy with Stat. Mech.

(1) analog of "local observables": functions $O(u)$ which depend only on $u_{\mathbf{k}}$ with $|\mathbf{k}| < K$. "Locality in momentum"
(2) analog of "Volume": just the cut-off N confining the $\mathbf{k}$
(3) analog of the "state parameter": the viscosity $\nu = \frac{1}{R}$ (irrev. case) or the enstrophy En (rev. case) (or energy E).

Equivalence is **conjectured** at $N = \infty$ corresponding to the Thermodynamic limit $V \to \infty$, for **all** R.

Averages of large scale observables will tend to the same values as $N \to \infty$ for $\mu_R^{irr,N} \in \mathcal{E}^{irr,N}$ of NS_{irr} and for $\mu_{En}^{rev,N} \in \mathcal{E}^{rev,N}$ provided, $\mathcal{D}(\mathbf{u}) \stackrel{def}{=} \sum_{\mathbf{k}} \mathbf{k}^2 |\mathbf{u}_{\mathbf{k}}|^2$ is s.t.

$$\mu_R^{irr,N}(\mathcal{D}) = En, \qquad \text{or} \qquad \mu_{En}^{rev,N}(\alpha) = \frac{1}{R}$$

Remark that multiplying the NS eq. by $\overline{u}_{\mathbf{k}}$ and sum on $\mathbf{k}$:

$$\frac{1}{2}\frac{d}{dt}\sum_{\mathbf{k}}|u_{\mathbf{k}}|^2 = -\gamma\mathcal{D}(\mathbf{u}) + W(\mathbf{u}), \quad \gamma = \nu \; or \; \alpha(\mathbf{u})$$

here the transport terms $= 0$, $D = 2, 3$,
$\mathcal{D}(\mathbf{u}) = \sum_{\mathbf{k}} \mathbf{k}^2 |\mathbf{u}_{\mathbf{k}}|^2 =$ enstrophy and $W = \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}} \mathbf{u}_{-\mathbf{k}} =$ work per unit time of the external force.
Hence time averaging

$$\frac{1}{R}\mu_R^{irr,N}(\mathcal{D}) = \mu_R^{irr,N}(W), \qquad \mu_{En}^{rev,N}(\alpha)En = \mu_{En}^{rev,N}(W)$$

But W is local (as $\mathbf{f}$ is such) and, if the conjecture holds, has equal average under the equivalence condition: hence $\mu_R^{irr,N}(\mathcal{D}) = En$ **implies** the relation

$$\lim_{N\to\infty} R\mu_{En}^{rev,N}(\alpha) = 1$$

This becomes a first rather stringent test of the conjecture.

Since the equivalence rests on the rapid fluctuations of $\alpha(u)$ a second idea is that if N is **kept finite** then, more generally, if O is a large scale observable it should be:

$$\mu_R^{irr,N}(O) = \mu_{En}^{rev,N}(O)(1+o(1/R)) \qquad \textbf{if} \qquad \mu_R^{irr,N}(\mathcal{D}) = En$$

So a different idea arises. In many phenomenological and dissipative equations of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \nu\mathbf{x} + \mathbf{g}$ the ν can be replaced by $\alpha(\mathbf{x})$ so that $E = \mathbf{x}^2$=const.=
If for $\nu = 0, \mathbf{g} = \vec{0}$ the motion is strongly chaotic then

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \nu\mathbf{x} + \mathbf{g},$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \alpha(\mathbf{x})\mathbf{x} + \mathbf{g}, \qquad \alpha(\mathbf{x}) = \frac{\mathbf{g}\cdot\mathbf{x}}{\mathbf{x}^2}$$

Equivalence if $\nu \to 0$ between stationary μ_ν^{irr} and μ_E^{rev} if

$$\mu_\nu^{irr}(\alpha) = E$$

What is special to NS to conj. that $R \to \infty$ is **not** needed?

It is its being a scaling limit of a microscopic equation whose evolution is certainly chaotic and reversible.

Therefore NS is **different** from the many phenomenological and dissipative equations which are not directly related to fundamental equations.

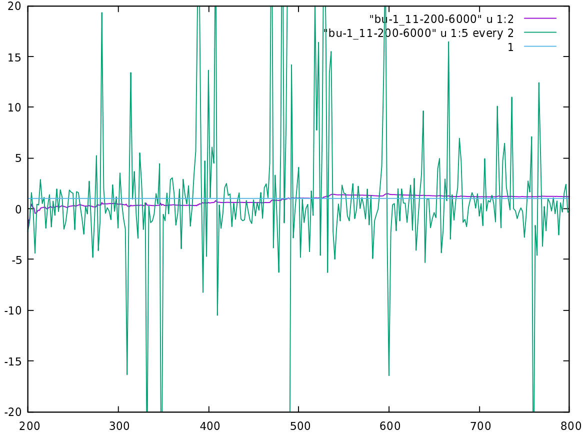
For the latter cases strong chaos is necessary if a friction parameter is changed into a fluctuating quantity.
There are many examples of phenomenological equations

(1) (highly) truncated NS equations ($N < \infty$ fixed), [4],
(2) NS with Ekman friction ($-\nu\vec{u}$ instead of $\nu\Delta\vec{u}$), [5, 6],
(3) Lorenz96 model, [7],
(4) Shell model of turbulence, (GOY), [8]

In such equations $R \to \infty$ is necessary: and, for each of them, it has been tested in few cases.

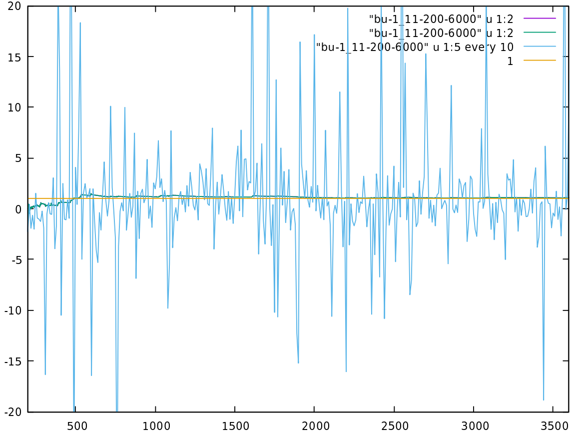
But it will be useful to pause to illustrate a few prelimnary simulations and checks.

Unfortunately the simulations are in dimension 2 ($D = 3$ is at the moment beyond the available (to me) computational tools) although present day available NS codes should be perfectly capable to perform detailed checks in rapid time.



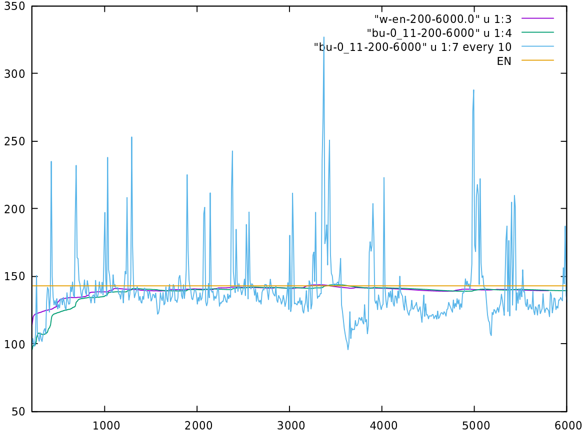
FigA32-19-17-11.1-detail

Fig.1-dettaglio: Running average of reversible friction $R\alpha(u) \equiv R\frac{2Re(f_{-\mathbf{k}_0} u_{\mathbf{k}_0})\mathbf{k}_0^2}{\sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2}$, superposed to conjectured 1 and to the fluctuating values of $R\alpha(u)$. Initial transient is clear. Evol.: NS_{rev}, **R=2048**, 224 modes, Lyap. $\simeq 2$, x-unit $= 2^{19}$



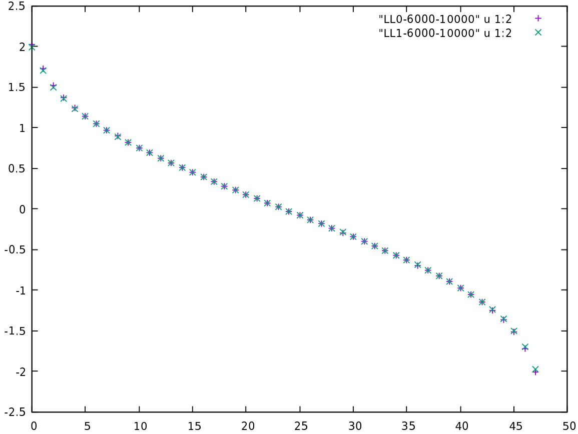
FigA32-19-17-11.1-all

Fig.1: As previous fig. but time 8 times longer: data reported "every 10", **or** black.



FigEN32-19-17-11.1

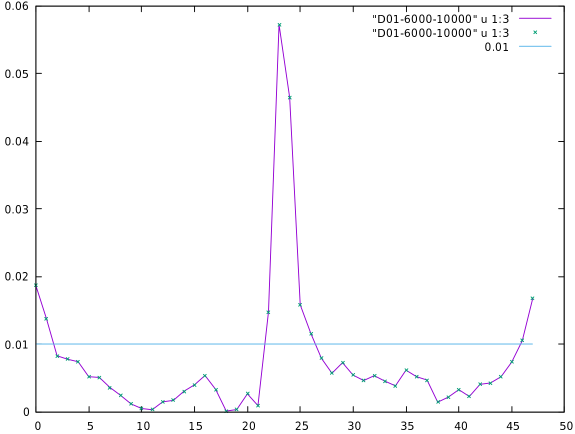
Fig.2: NS_{irr}: **Running** average of the work $R\sum_{\mathbf{k}} F_{-\mathbf{k}} u_{\mathbf{k}}|$ (**violet**) in NS_{rev}; and **convergence** to average enstrophy En (**orange** straight line),
blue is running average of enstrophy $\sum_{\mathbf{k}} \mathbf{k}^2 |u_{\mathbf{k}}|^2$ in NS_{irr}, enstrophy **fluctuations** violet in NS_{irr}: **R=2048**.



FigL16-19-17-11.01

Fig.3: Spectrum (**local**) Lyapunov V=48 modes reversible & irreversible superposed; **R=2048**.

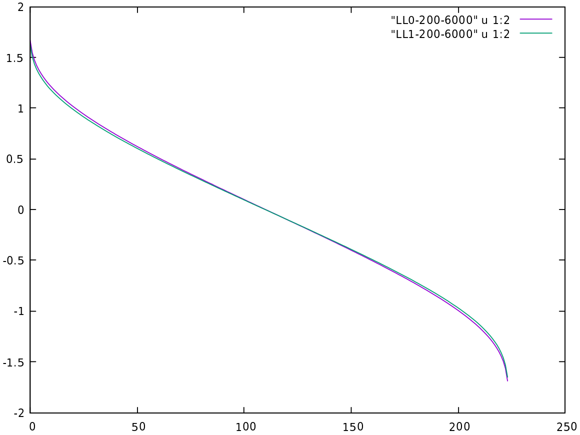
The difference can be made visible as:



FigDiff16-191711-01

Fig.4: **Relative Difference** of (local) Lyap. exponents in Fig. preced. R=2048, 48 modes.

Graph of $\frac{|\lambda_k^{rev} - \lambda_k^{irr}|}{\max(|\lambda_k^{rev}|, \lambda_k^{irr}|)}$; Level line marks 1%.



FigL32-19-17-11.01

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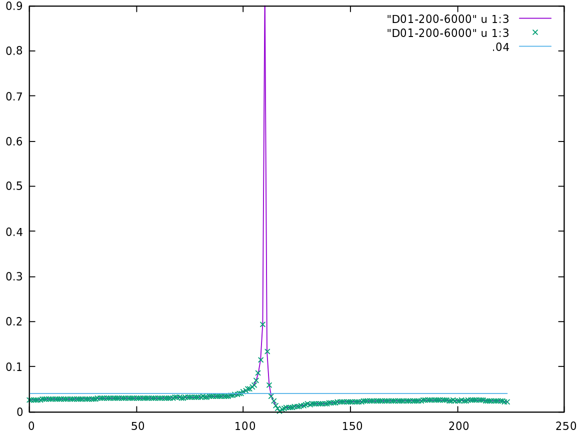
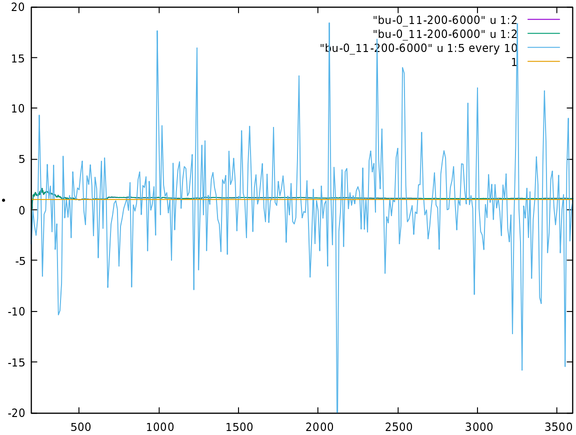
Fig.5: More Lyapunov spectrume in 15×15 modes (i.e. for NS2D rever. & irrev. $R = 2048$, 240 modes on 2^{13} steps. Spectra evalued every 2^{19} integr. steps. (and averaged over 200 samples).



FigDiff32-19-17-11.01

Fig.6: **Relative difference** of the (local) Lyapunov exp. of the preceding fig. 240 modes. The line is the 4% **level**.

The following Fig.7 (similar to Fig.1 but w. NS_{irr}):



FigA32-19-17-11.0-all

Fig.7: As Fig.1 but running average of reversible friction $R\alpha(\mathbf{u})$ regarded as observ. in NS_{irr}, superposed ro value 1 and to fluctuating values of $R\alpha(\mathbf{u})$. An extension of conjecture since $\alpha(\mathbf{u})$ is not local.

The figure suggests (from the theory of Anosov systems):
(1) **Check** the "Fluctuation Relation" in the **irreversible** evollution: for the divergence (trace of the Jacobian) $\boldsymbol{\sigma}(u) = -\sum_{\mathbf{k}} \partial_{u_{\mathbf{k}}} (\dot{u}_{\mathbf{k}})_{rev}$: let p (time τ average of $\frac{\sigma}{\langle \sigma \rangle}$)

$$p \stackrel{def}{=} \frac{1}{\tau} \int_0^\tau \frac{\boldsymbol{\sigma}(\mathbf{u}(t))}{\langle \boldsymbol{\sigma} \rangle_{irr}} dt,$$

then a theorem for Anosov systems:

$$\frac{P_{srb}(p)}{P_{srb}(-p)} = e^{\tau \, \mathbf{1} \, \mathbf{p} \, \langle \boldsymbol{\sigma} \rangle_{\mathbf{irr}}} \text{ (sense of large deviat. as } \tau \to \infty)$$

it is a *"reversibility test on the irreversible flow"*

Anosov systems play the role, in chaotic dynamics that harmocic oscillators cover for ordered motions.
They are a paradigm of chaos.

The idea is based on **Sinai** (for Anosov syst.), **Ruelle, Bowen** (for Axioms A syst.),[9, 10, 11]

Attention on Anosov syst. leads to:

Chaotic hypothesis: *An empirically chaotic evolution takes eventually place on a smooth surface $\mathcal{A}$, "attracting surface" in phase space and, on $\mathcal{A}$, the evolution (map S or flow S_t) is a Anosov syst.*

It is a strict and general heuristic interpretation of the original ideas on turbulence phenomena, [11], see [12, endnote 18], [13, 14], [15].

BUT: various are the obstacles to its applicability and resolving them leads to new interesting problems.

Problem: if $\mathcal{A} \subset M_V$ e $\mathcal{A}$ has lower dimension, the time reversal symmetry I cannot be applied because $I\mathcal{A} \neq \mathcal{A}$. This certainly occurs if V becaomes large enough, [16, 17].

However a further symmetry P may exist between $\mathcal{A}$ and $I\mathcal{A}$ *commuting* with evolution S_t: $PS_t = S_tP$.

Then $P \circ I : \mathcal{A} \to \mathcal{A}$ becomes a time reversal symmetry of the motion restricted to $\mathcal{A}$. And there are geometrical conditions which in special cases guarantee existence of P ("Axiom C" systems, [18]).

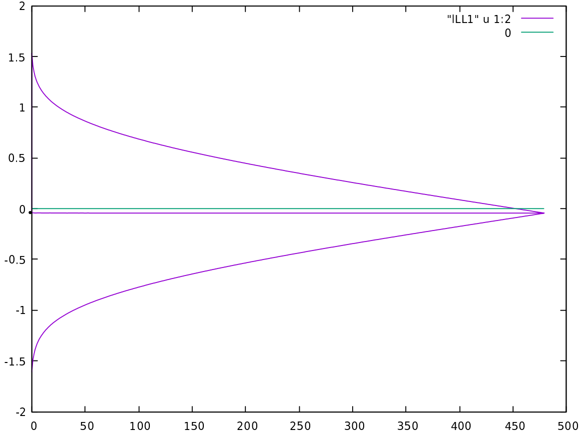
However even supposing existence of P, still **is is not** possible to apply FR because, at best, it would concern the contraction $\sigma_{\mathcal{A}}(\mathbf{u})$ of $\mathcal{A}$ and not the $\sigma(\mathbf{u})$ of M_V.

The $\sigma(\mathbf{u})$ riceives contributions from the exponential approach to $\mathcal{A}$: which obviously do not contribute to $\sigma_{\mathcal{A}}$. How to recognize such contributions ?

Help could come from “pairing rule”
Often the Lyapunov exponents (local and global) arise in pairs with almost constant average or average on a regular curve.

In several systems the pairs have an exactly constant average.

An idea can be obtained from the local exponents (the eigenvalues of the simmetric part of the Jacobian matrix of the evolution).
For instance in NS it is



FIGll-64-19-17-11

Fig.8: $R = 2048$, **960modes**, **local** exponents ordered decreasing: s.t. λ_k, $0 \le k < d/2$,
and increasing $\lambda_{d-k}, 0 \le k < d/2$,
the line $\frac{1}{2}(\lambda_k + \lambda_{d-1-k})$ and the line $\equiv 0$. Irreversible case and apparent pairing rule

The graph of the reversible exponents is again almost superposed to the above and the following figure gives the relative difference of the 960 correponding exponents.

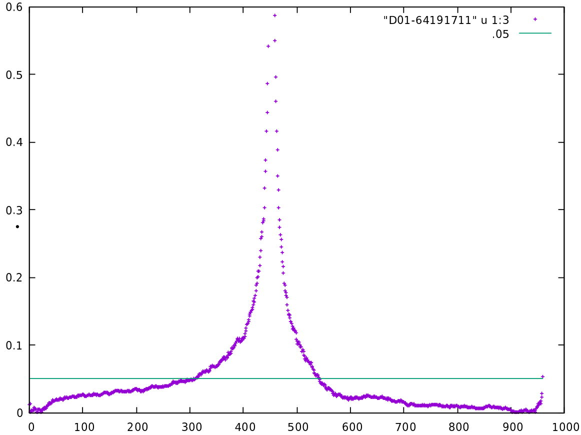


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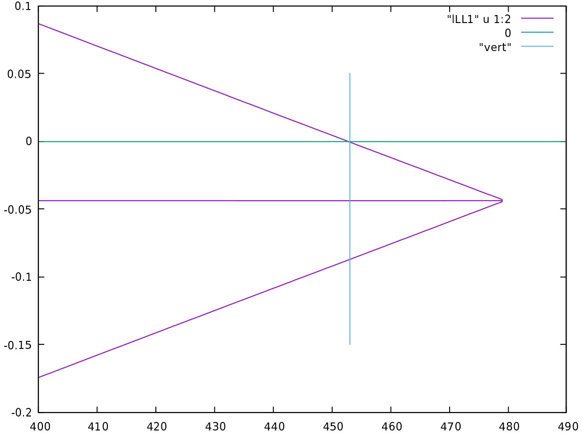
Fig.9: Relative difference $\frac{|\lambda_k^{rev}-\lambda_k^{irr}|}{\max(|\lambda_k^{rev}|,|\lambda_k^{irr}|)}$ between reversible and irreversible local exp. in Fig.7. Line = 4% **level**.



. FIGll-detail64-19-17-11

Fig.10: Detail of Fig.8 showing the NS_{irr} exponents and the line $\equiv 0$: it illustrates the"dimensional loss" $\sim \frac{450}{490}$.
$R = 2048$, 960 modes.

The figures indicate:

(a) revers. and irrrev. exponents are very close: but this **does not follow** from the conject. (as the exponents are not local observables) $\rightarrow$ suggests: possible equivalence for a larger class of observables.

(b) It has been proposed that the attracting surface $\mathcal{A}$ has dimension = twice the number of positive exponents: which implies in cases of pairing that it is twice the number of pairs with opposite sign.

Implication: $\sigma_{\mathcal{A}}(\mathbf{u})$ is proportional to the total $\sigma(\mathbf{u})$ in the cases of pairing to a constant

$$\sigma_{\mathcal{A}}(\mathbf{u}) = \boldsymbol{\varphi}\sigma(\mathbf{u}), \qquad \boldsymbol{\varphi} = \frac{number\ of\ opposite\ pairs}{total\ number\ of\ pairs}$$

and in the case of pairing to a more general curve $\sigma_{\mathcal{A}}(u) = \sigma(u) + \sum_{pairs<0}(\lambda_j + \lambda'_j)$. Why?

Idea: negative pairs correspond to the exponents associated with the attraction to $\mathcal{A}$: hence do not count for the computation of $\sigma_{\mathcal{A}}$.
The FR will hold, by the C.H., but with a slope $\boldsymbol{\varphi} < 1$:

$$\tau p \boldsymbol{\varphi} \sigma, \quad \text{rather than} \quad \tau p \sigma : \quad \text{in fig. } \boldsymbol{\varphi} \simeq \frac{\mathbf{450}}{\mathbf{490}}$$

If true: this will be a check of reversibility in NS_{irr}.

IF FR holds, it is possible to think to one more statistical ensemble $\mathcal{E}^{st}$ consisting in the stationary PDF's for NS_{st}

$$\dot{u}_\alpha = -(\vec{u}\cdot\boldsymbol{\partial})u_\alpha - \partial_\alpha p + \nu(u)\Delta u_\alpha + F_\alpha, \qquad \partial_\alpha u_\alpha = 0$$

where $\nu(u)$ is a stochastic process (*e.g.* gaussian) uncorrelated but with average $\langle\, \nu \,\rangle = \frac{1}{R}$ and with a distribution respecting the FR (*i.e.* dispersion = average in the gaussian case).

More elaborate checks are being attempted:

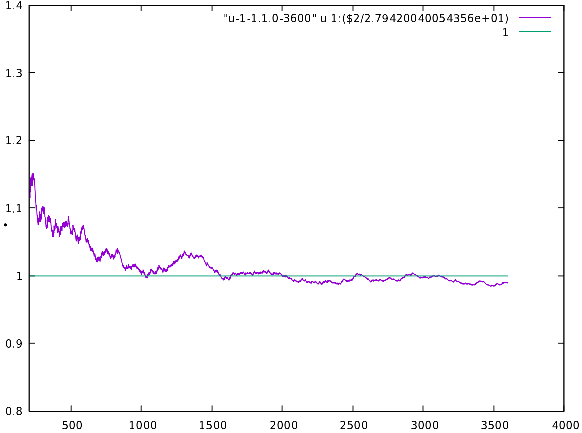
(a) moments of large scale observables rev & irr

(b) local Lyap. exponents of other matrices different from the Jacobiank

(c) check of the fluctuation rel., particularly in the irrev. cases, which from the previous figures is shown to be accessible already with 960 modes and $R = 2048$: $\Rightarrow$ FR with slope $\varphi < 1$ (Axiom C ?), [14, 13].

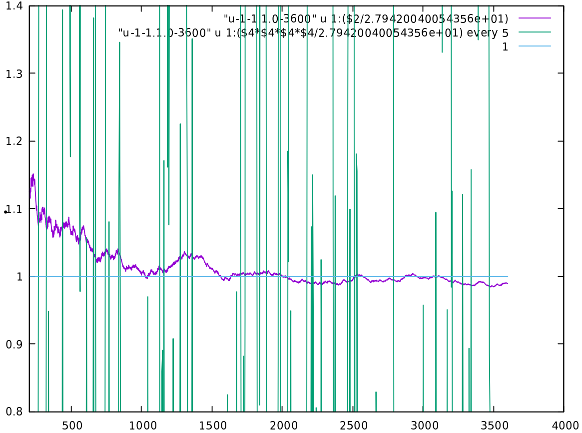
(d) More values of R and N an interesting example is Fig.10 with R much larger than in the preceding cases.

Example of moments of local observables:



FIGu0-64-191711-10

Fig.11: Running averages rev of $|Re\,u_{11}|^4/\langle\, Re\,u_{11}|^4\,\rangle_{irr}$, $R = 2048$, 224 modes. Conjecture yields ratio tending to 1



FIGu1-64-191711-10

Fig.12: Same running averages rev of $|Re\, u_{11}|^4 / \langle\, Re\, u_{11}|^4\,\rangle_{irr}$, for $R = 2048$, and their rev. fluctuations, 224 modes.

Finally a rigorous estimate of the number $\mathcal{N}$ of Lyap. exp. (local and global ordered decreasing), needed so that their sum remains > 0:

$$\lesssim \sqrt{2} A (2\pi)^2 \sqrt{R} \sqrt{R\, En},\, A = 0.55..$$

in dimension 2, while at dimension 3 a similar estimate holds but it involves a norm different from the enstrophy. (Ruelle if $d = 3$ and Lieb if $d = 2, 3$, [19, 17]. Applied here it would give $\mathcal{N} \sim 2.10^4$ for NS 2D: not accessible in the simulations presented here but not impossible in principle with available computers and computation methods already available, at least if $D = 2$.

Finally a warning that further careful checks are required, particularly because the inspiring ideas are, to say the least, **controversial** as shown by the following quote, selected among the several, from a well known treatise:

CH is dismissed (by many) with arguments like (1999)

'More recently Gallavotti and Cohen have emphasized the "nice" properties of Anosov systems. Rather than finding realistic Anosov examples they have instead promoted their "Chaotic Hypothesis": if a system behaved "like" a [wildly unphysical but well-understood] time reversible Anosov system there would be simple and appealing consequences, of exactly the kind mentioned above. Whether or not speculations concerning such hypothetical Anosov systems are an aid or a hindrance to understanding seems to be an aesthetic question., [20].

Avoiding to comment on the statement I stress that Statistical Mechanics, from Clausius, Boltzmann and Maxwell has been a simple, surprising, consequence of the **"[wildly unphysical but well-understood]"** periodicity of the collective motions of 10^{19} gas molecules, [21].

Quoted references

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