Statistical ensembles out of equilibrium: turbulence

Equilibrium states $\longleftrightarrow$ different probability distributions, *e.g.* canonical or microcanonical:

Reminder:

ρV particles in volume $V \Rightarrow$ families $\mathcal{E}^{mc}, \mathcal{E}^{c}, \ldots$ of distributions; elements "parameterized" by E, β,...

1) "observables of interest": local observables $O \in \mathcal{O}_{loc}$: $O(\mathbf{p},\mathbf{q})$, depend on $q_i \in \mathbf{q}$ with $q_i \in \Lambda$, Λ = volume $\ll V$

2) distributions $\mu_\beta^V \in \mathcal{E}^c$ and $\widetilde{\mu}_E^V \in \mathcal{E}^{mc}$ are **correspondent** if β, E are s.t.

$$\mu_\beta^V(H_V(\mathbf{p},\mathbf{q})) = E \;\Rightarrow\; \lim_{V\to\infty} \mu_\beta^V(O) = \lim_{V\to\infty} \widetilde{\mu}_E^V(O)$$

and μ's are "*equivalent* in the thermodynamic limit".

Is it possible a similar description of the stationary states of nonequilibrium systems?

Evolution eq. of $\mathbf{u}$ on "phase space" $\mathbf{M}$ (∞-dim.) depending on a parameter R:

$$\dot{\mathbf{u}} = \mathbf{f}_{\mathbf{R}}(\mathbf{u}) \qquad \text{(formally)}$$

"Difficult":even existence-1-qness **open**.

In any theory of large (macroscopic) systems theories are based on two key aspects

(1) Regularization of equations (via a "cut off")

(2) Restriction on observables ("local observables")

Regularization, necessary in essentially all cases, replaces $f_R(\vec{u})$ (∞-dim) by a regularized $f_R^V(\vec{u})$ (finite dimensional).

Stationary distrib. $\mu_R^V(du)$ will be uniquely determined by Ruelle's extension of ergodic hypothesis (*i.e.* SRB distrib.).

Form a family $\mathcal{E}_R^V$ of distributions assigning average values to the restricted observables.

(a) Stat. Mech: looks at local observables and cut-off $V =$ container size; $\Rightarrow$ find their averages at limit as $V \to \infty$:

(b) Fluid Mech.: looks at large scale onservables (*i.e.* functions of velocities with "waves" $|\mathbf{k}| < K \ll N$) and cut-off N on the maximum wave $|\mathbf{k}|$: $\Rightarrow$ find averages at limit as $N \to \infty$

Once physical observables are restricted, it is expected (?) that several equations could describe stationary states of the same system.

E.g. hard core balls are described by Hamilton eq.s but also by the isothermal equations, [1],

$$\dot{\mathbf{q}} = \mathbf{p}, \qquad \dot{\mathbf{p}} = -\boldsymbol{\partial}_{\mathbf{q}} V(\mathbf{q}) - \alpha(\mathbf{p},\mathbf{q})\mathbf{p}$$

where $\alpha(\mathbf{p},\mathbf{q})$ is a multiplier which imposes $T(\mathbf{p}) = const.$

Stationary states of the two equations are parameterized by energy E or kinetic energy $k_B T = \beta^{-1}$ and will be

$$\mu_E^{mc,V} = \delta(H(\mathbf{p},\mathbf{q}) - E)d\mathbf{p}d\mathbf{q} \qquad or, respectively:$$

$$\mu_\beta^{c,V} = e^{-\beta_0 V(\mathbf{q})}\delta(T(\mathbf{p}) - N\beta^{-1})d\mathbf{p}d\mathbf{q}, \qquad \beta_0 = \beta(1 - \frac{1}{3N})$$

Equivalent (on local onservables) if
$\mu_\beta^{c,V}(H) = E \;\Rightarrow\; \lim_{V\to\infty} \mu_\beta^{c,V}(O) = \lim_{V\to\infty} \mu_E^{mc,V}(O)$.

Interesting cases arise when equations obey a fundamental symmetry but may be phenomenologically described by non symmetric equations (spontaneously broken symmetry).

Since a fundamental symmetry cannot be broken it is to be expected that the same system can be described equally well by symmetric eqs. (equivalent on special observables).

Consider, as a typical case, the Navier-Stokes equations.

In incompressible fluid can be regarded as Euler equations subject to a thermostat adapting the pressure to the heat due to the viscosity: it turns the equations into time-reversal breaking ones.

Paradigmatic case is periodic NS fluid, [2, 3],

(a) 2/3-Dim., incompressible,

(b) **fixed large scale forcing** F (*e.g.* **with only one or few** waves and $\|F\|_2 = 1$),

(c) with thermostat. to dissipate heat via viscosity $\boldsymbol{\nu} = \frac{\mathbf{1}}{\mathbf{R}}$ (consistently $p = P(\tau, T)$).

NS_{irr}: $\dot{u}_\alpha = -(\vec{u} \cdot \boldsymbol{\partial}) u_\alpha - \partial_\alpha p + \frac{1}{R} \Delta u_\alpha + F_\alpha, \qquad \partial_\alpha u_\alpha = 0$

Velocity: $\vec{u}(x) = \sum_{\vec{k} \neq \vec{0}} u_{\mathbf{k}} \frac{i\mathbf{k}^\perp}{|\mathbf{k}|} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \overline{u}_{\mathbf{k}} = u_{-\mathbf{k}}$ (NS-2D)

$NS_{2,irr}$: $\dot{u}_{\mathbf{k}} = \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} \frac{(\mathbf{k}_1^\perp \cdot \mathbf{k}_2)(\mathbf{k}_2^2 - \mathbf{k}_1^2)}{2|\mathbf{k}_1||\mathbf{k}_2||\mathbf{k}|} u_{\mathbf{k}_1} u_{\mathbf{k}_2} - \nu \mathbf{k}^2 u_{\mathbf{k}} + f_{\mathbf{k}}$

$I u_\alpha = -u_\alpha$ implies $I S_t^{irr} \neq S_{-t}^{irr} I$, $\Rightarrow$: irreversibility.

"Regularize eq.": waves $|\mathbf{k}_j| \leq N$. At UV-Cut-off , N.

Given init. data u, evolution $u \to S_t^{irr} u$ generates a steady state (*i.e.* a SRB probability distr.) $\mu_R^{irr,N}$ on M_N.

Unique out a volume 0 of u's, for simplicity [AT R small: "NS gauge symmetry" exists.; phase transitions, [4, 5, 6].

As R varies steady distr. $\mu_R^{irr,N}(du)$ are collected in $\mathcal{E}^{irr,N}$:

A statistical ensemble of stationary nonequilibrium distrib. for NS_{irr}.

Average energy E_R, average dissipation En_R, Lyapunov spectra (local and global) ... will be defined, *e.g.*:

$$E_R = \int_{M_N} \mu_R^{irr,N}(du)||u||_2^2, \qquad En_R = \int_{M_N} \mu_R^{irr,N}(du)||\mathbf{k}u||_2^2$$

Consider new equation, NS_{rev} (with cut-off N):

$$\dot{\mathbf{u}}_{\mathbf{k}} = \sum_{\mathbf{k_1}+\mathbf{k_2}=\mathbf{k}} \frac{(\mathbf{k}_1^{\perp}\cdot\mathbf{k_2})(\mathbf{k}_2^2-\mathbf{k}_1^2)}{2|\mathbf{k_1}||\mathbf{k_2}||\mathbf{k}|}\mathbf{u}_{\mathbf{k_1}}\mathbf{u}_{\mathbf{k_2}} - \alpha(\mathbf{u})\mathbf{k}^2\mathbf{u}_{\mathbf{k}} + f_{\mathbf{k}}$$

with α **s. t.** $\mathcal{D}(u) = ||\mathbf{k}u||_2^2 = En$ (the enstrophy)is **exact const of motion** on $u \to S_t^{rev}u$.:

$$\Rightarrow\ \alpha(u) = \frac{\sum_{\mathbf{k}}\mathbf{k}^2 F_{-\mathbf{k}}u_{\mathbf{k}}}{\sum_{\mathbf{k}}\mathbf{k}^4|u_{\mathbf{k}}|^2} \qquad e.g.\ \ D=2$$

New eq. is reversible: $IS_t^{rev}u = S_{-t}^{rev}Iu$ (as α is odd).

α is "a reversible viscosity"; (if $D=3$ α is ∼different)

Rev. eq. can be empirical model of "thermostat" on the fluid and should (?) have **same effect** of empirical constant friction.

NS_{rev} generates a family of steady states $\mathcal{E}^{rev,N}$ on M_N: $\mu^{rev,N}_{En}$ parameterized by constant value of **enstrophy** En.

$\alpha(u)$ in NS_{rev} **will wildly fluctuate** at large R (*i.e.* small viscosity ν) thus "**self averaging**" to a const. value ν "homogenizing" the eq. into NS_{irr} with viscosity ν.

Of course could impose multiplier [7, 8] $\alpha'(u) = \frac{\sum_{\mathbf{k}} f_{\mathbf{k}}\overline{u}_{\mathbf{k}}}{\sum_{\mathbf{k}} |\mathbf{k}|^2|u_{\mathbf{k}}|^2}$: it would fix energy $E = \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2$.

Equivalence mechanism by analogy with Stat. Mech.

(1) analog of "local observables": functions $O(u)$ which depend only on $u_{\mathbf{k}}$ with $|\mathbf{k}| < K$. "Locality in momentum"

(2) analog of "Volume": just the cut-off N confining the $\mathbf{k}$

(3) analog of "state parameter": viscosity $\nu = \frac{1}{R}$ (irrev. case) or enstrophy En (rev. case) (or energy E ?).

$$\text{Equivalence condition}: \ \mu^{rev,N}_{En}(\alpha) = \frac{1}{R}$$

Equivalence is **conjectured** at $N = \infty$ corresponding to the Thermodynamic limit $V \to \infty$, for **all** R.

Averages of large scale observables will tend to the same values as $N \to \infty$ for $\mu^{irr,N}_R \in \mathcal{E}^{irr,N}$ of NS_{irr} and for $\mu^{rev,N}_{En} \in \mathcal{E}^{rev,N}$ **provided**, $\mathcal{D}(\mathbf{u}) \stackrel{def}{=} \sum_{\mathbf{k}} \mathbf{k}^2|\mathbf{u}_{\mathbf{k}}|^2$ is s.t.

$$\mu^{irr,N}_R(\mathcal{D}) = En, \qquad \text{or} \qquad \mu^{rev,N}_{En}(\alpha) = \frac{1}{R} = \nu$$

Balance: multiplying NS eq. by $\overline{u}_{\mathbf{k}}$ and sum on $\mathbf{k}$:

$$\frac{1}{2}\frac{d}{dt}\sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 = -\gamma\mathcal{D}(\mathbf{u}) + W(\mathbf{u}), \quad \gamma = \nu \ or \ \alpha(\mathbf{u})$$

(transport terms $= 0$, $D = 2, 3$), $\mathcal{D}(\mathbf{u}) = \sum_{\mathbf{k}} \mathbf{k}^2|\mathbf{u}_{\mathbf{k}}|^2 =$ enstrophy and $W = \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}}\mathbf{u}_{-\mathbf{k}} =$ power of external force.

Hence time averaging

$$\frac{1}{R}\mu_R^{irr,N}(\mathcal{D}) = \mu_R^{irr,N}(W), \qquad \mu_{En}^{rev,N}(\alpha)En = \mu_{En}^{rev,N}(W)$$

But W is local (as $\mathbf{f}$ is such) and, if the conjecture holds, has equal average under the equivalence condition: hence $\mu_R^{irr,N}(\mathcal{D}) = En$ **implies** the relation

$$\lim_{N\to\infty} R\mu_{En}^{rev,N}(\alpha) = 1$$

This becomes a first rather stringent test of the conjecture.

Since the equivalence rests on the rapid fluctuations of $\alpha(u)$ a second idea is that if N is **kept finite** then, more generally, if O is a large scale observable it should be:

$$\mu_R^{irr,N}(O) = \mu_{En}^{rev,N}(O)(1+o(1/R)) \qquad \textbf{if} \qquad \mu_R^{irr,N}(\mathcal{D}) = En$$

So a different idea arises.

In dissipative equations of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \nu\mathbf{x} + \mathbf{g}$ the ν can be replaced by $\alpha(\mathbf{x})$ so that $E = \mathbf{x}^2$=const.
If for $\nu = 0, \mathbf{g} = \vec{0}$ the motion is strongly chaotic then

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \nu\mathbf{x} + \mathbf{g},$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \alpha(\mathbf{x})\mathbf{x} + \mathbf{g}, \qquad \alpha(\mathbf{x}) = \frac{\mathbf{g}\cdot\mathbf{x}}{\mathbf{x}^2}$$

Equivalence if $\nu \to 0$ between stationary μ_ν^{irr} and μ_E^{rev} if

$$\mu_\nu^{irr}(\alpha) = E$$

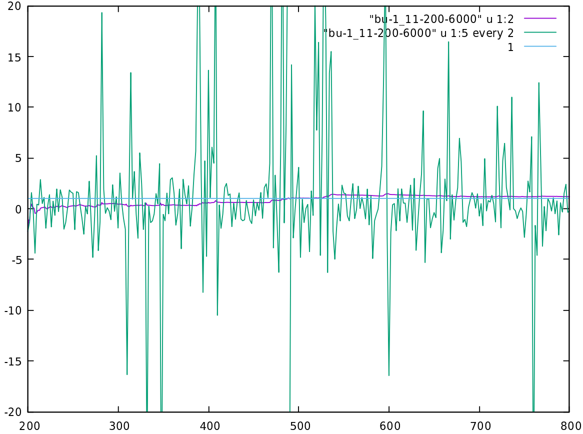
What is special to NS to conj. that $R \to \infty$ is **not** needed? It is its being a scaling limit of a microscopic equation whose evolution is certainly chaotic and reversible.

NS **differs** from phenomenological and dissipative equations not directly related to fundamental equations.

For the latter cases strong chaos is necessary if a friction parameter is changed into a fluctuating quantity.

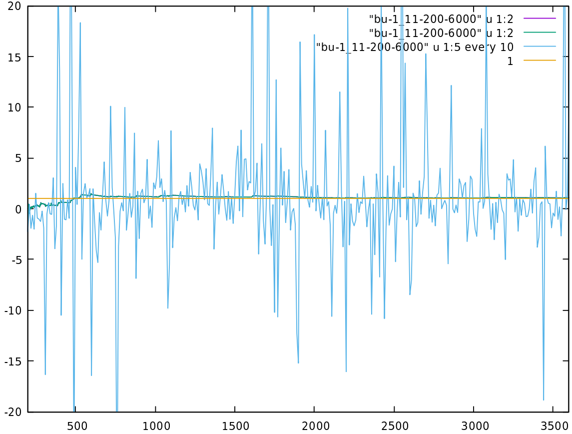
But it will be useful to pause to illustrate a few prelimnary simulations and checks.

Unfortunately the simulations are in dimension 2 ($D = 3$ is at the moment beyond the available (to me) computational tools) although present day available NS codes should be perfectly capable to perform detailed checks in rapid time, [8].



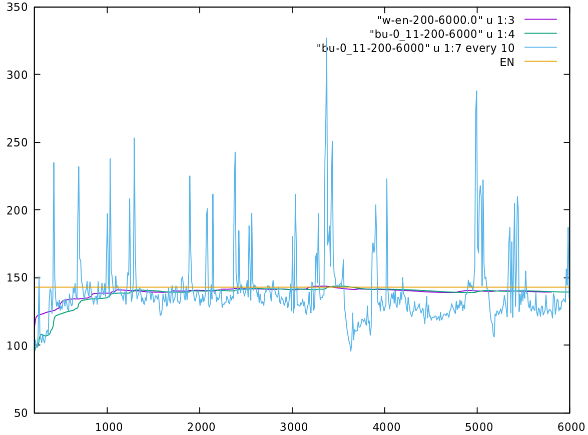
FigA32-19-17-11.1-detail

Fig.0 (detail): Running average of reversible friction $R\alpha(u) \equiv R\frac{2Re(f_{-\mathbf{k}_0} u_{\mathbf{k}_0})\mathbf{k}_0^2}{\sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2}$, superposed to conjectured 1 and to the fluctuating values of $R\alpha(u)$. **Initial transient $t < 800$.**
Evol.: NS_{rev}, **R=2048**, 224 modes, Lyap. $\simeq$ 2, x-unit = 2^{19}



FigA32-19-17-11.1-all

Fig.1: As previous fig. but time 8 times longer: data reported "every 10", **or** black.

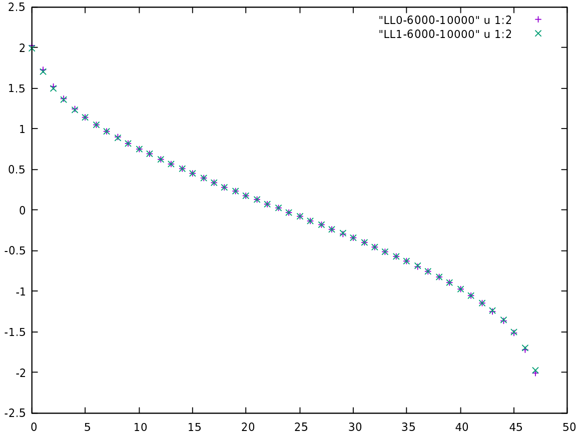


FigEN32-19-17-11.1

Fig.2: NS_{irr}: **Running** average of the work $R\sum_{\mathbf{k}} F_{-\mathbf{k}} u_{\mathbf{k}}|$ (**violet**) in NS_{rev}; and **convergence** to average enstrophy En (**orange** straight line),

blue is running average of enstrophy $\sum_{\mathbf{k}} \mathbf{k}^2 |u_{\mathbf{k}}|^2$ in NS_{irr}, enstrophy **fluctuations** violet in NS_{irr}: **R=2048**.

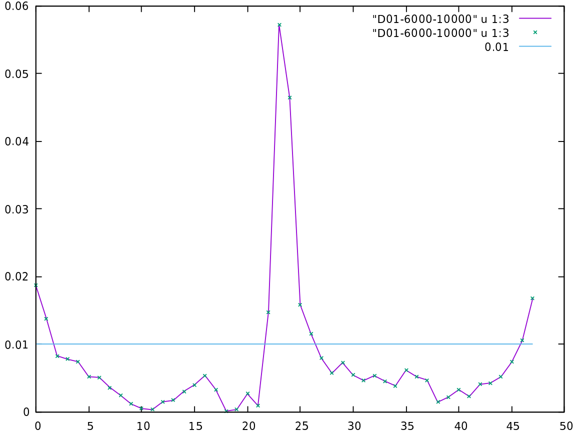
unexpected ?, [7]:



FigL16-19-17-11.01

Fig.3: Spectrum (**local**) Lyapunov V=48 modes reversible & irreversible superposed; **R=2048**.

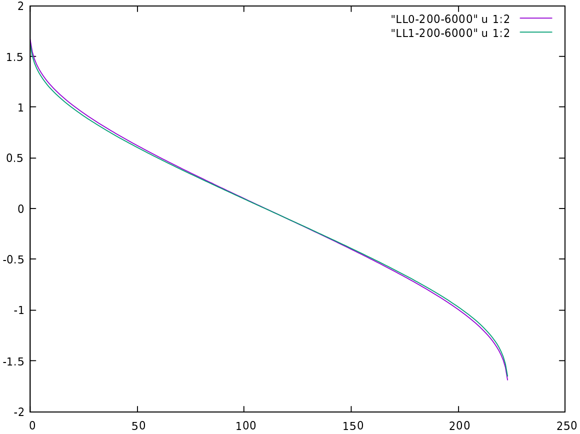
The difference can be made visible as:



FigDiff16-191711-01

Fig.4: **Relative Difference** of (local) Lyap. exponents in Fig. preced. R=2048, 48 modes.

Graph of $\frac{|\lambda_k^{rev} - \lambda_k^{irr}|}{\max(|\lambda_k^{rev}|, \lambda_k^{irr}|)}$; Level line marks 1%.



FigL32-19-17-11.01

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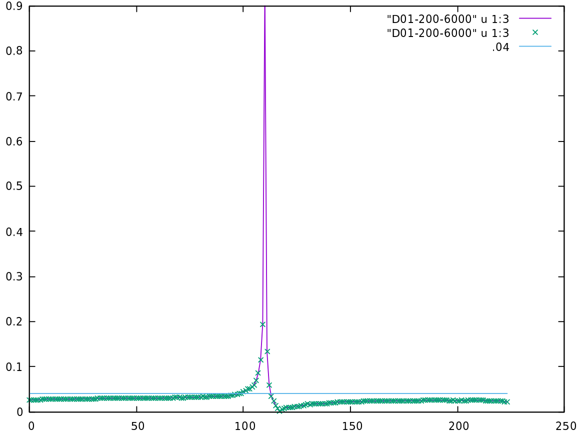
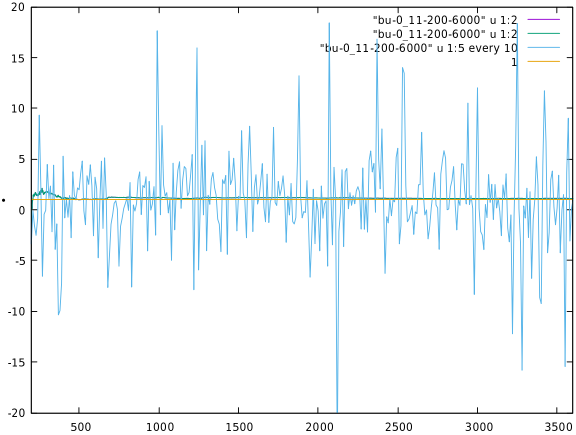
Fig.5: More Lyapunov spectrum in 15×15 modes (i.e. for NS2D rever. & irrev. $R = 2048$, 240 modes on 2^{13} steps. Spectra evalued every 2^{19} integr. steps. (and averaged over 200 samples).



FigDiff32-19-17-11.01

Fig.6: **Relative difference** of the (local) Lyapunov exp. of the preceding fig. 240 modes. The line is the 4% **level**.

The following Fig.7 (similar to Fig.1 but w. NS_{irr}):



FigA32-19-17-11.0-all

Fig.7: As Fig.1 but running average of reversible friction $R\alpha(\mathbf{u})$ regarded as observ. in NS_{irr}, superposed ro value 1 and to fluctuating values of $R\alpha(\mathbf{u})$. An extension ? of conjecture since $\alpha(\mathbf{u})$ is not local.

The figure suggests (from the theory of Anosov systems):
Check the "Fluctuation Relation" in the **irreversible** evolution: for the divergence (trace of the Jacobian) $\boldsymbol{\sigma}(u) = -\sum_{\mathbf{k}} \partial_{u_{\mathbf{k}}} (\dot{u}_{\mathbf{k}})_{rev}$: let p (time τ average of $\frac{\sigma}{\langle \sigma \rangle}$)

$$p \stackrel{def}{=} \frac{1}{\tau} \int_0^\tau \frac{\boldsymbol{\sigma}(\mathbf{u}(t))}{\langle \boldsymbol{\sigma} \rangle_{irr}} dt,$$

then a theorem for Anosov systems:

$$\frac{P_{srb}(p)}{P_{srb}(-p)} = e^{\tau\,\mathbf{1}\,\mathbf{p}\,\langle \boldsymbol{\sigma} \rangle_{\mathbf{irr}}} \text{ (sense of large deviat. as } \tau \to \infty)$$

it is a *"reversibility test on the irreversible flow"*

Anosov systems play the role, in chaotic dynamics that harmocic oscillators cover for ordered motions. They are a paradigm of chaos. Are NS Anosov systems?

The idea is based on **Sinai** (for Anosov syst.), **Ruelle, Bowen** (for Axioms A syst.),[9, 10, 11] *Chaotic hypothesis.*

Can this be applied to turbulence **?** **However:**

Problem 1: if attracting set $\mathcal{A}$ has lower dimension, time reversal symmetry I cannot be applied because $I\mathcal{A} \neq \mathcal{A}$. This certainly occurs if N becomes large enough, [12, 13].

Help could come **if** exists further symmetry P between $\mathcal{A}$ and $I\mathcal{A}$ *commuting* with S_t: $PS_t = S_tP$.

Then $P \circ I : \mathcal{A} \to \mathcal{A}$ becomes a time reversal symmetry of the motion restricted to $\mathcal{A}$. And there are geometrical conditions which in special cases guarantee existence of P ("Axiom C" systems, [14]).

Problem 2: even supposing existence of P, still **is is not** possible to apply FR because, at best, it would concern the contraction $\sigma_{\mathcal{A}}(\mathbf{u})$ of $\mathcal{A}$ and not the $\sigma(\mathbf{u})$ of M_V.

The $\sigma(\mathbf{u})$ receives contributions from the exponential approach to $\mathcal{A}$: which obviously do not contribute to $\sigma_{\mathcal{A}}$.

How to recognize such contributions **?**

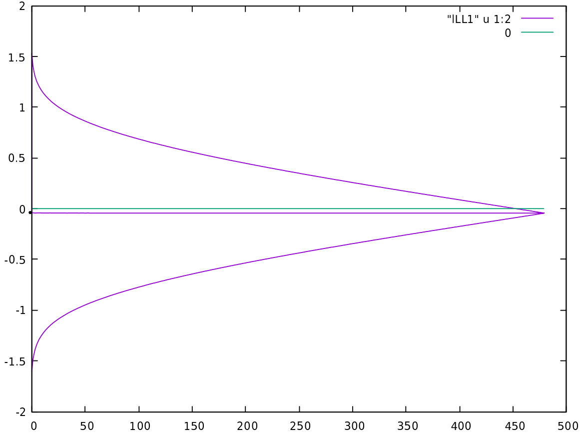
Help could come from "pairing rule"

Often Lyapunov exps (local and global) arise in pairs with almost constant average or average on a regular curve.

In a few systems pairs have an exactly constant average.

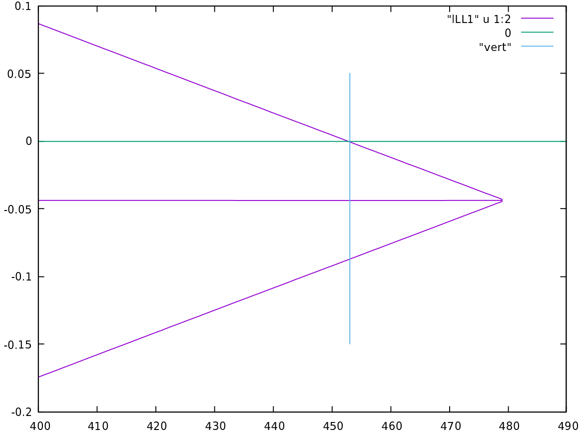
An idea can be obtained from the local exponents (eigenvalues of the symmetric part of the evolution Jacobian matrix).

For instance NS seems to enjoy a pairing rule:



FIGll-64-19-17-11

Fig.8: $R = 2048$, **960modes**, **local** exponents ordered decreasing: s.t. λ_k, $0 \le k < d/2$,
and increasing $\lambda_{d-k}, 0 \le k < d/2$,
the line $\frac{1}{2}(\lambda_k + \lambda_{d-1-k})$ and the line $\equiv 0$. Irreversible case and apparent pairing rule



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FIGll-detail64-19-17-11

Fig.9: Detail of Fig.8 showing the NS_{irr} exponents and the line $\equiv 0$: it illustrates the"dimensional loss" $\sim \frac{450}{490}$.
$R = 2048$, 960 modes.

The figures indicate:

(a) can check: revers. and irrrev. exps are very close: (but this **does not follow** from the conject. as exps are not local observables) → suggests: possible equivalence for a larger class of observables.

(b) It has been proposed, [15, p.445],[7], that attracting surface $\mathcal{A}$ dimension = twice the number of positive exponents: hence in cases of pairing it is twice num. of opposite sign pairs.

Implication: $\sigma_{\mathcal{A}}(\mathbf{u})$ is proportional to the total $\sigma(\mathbf{u})$ if pairing to a constant

$$\sigma_{\mathcal{A}}(\mathbf{u}) = \boldsymbol{\varphi}\sigma(\mathbf{u}), \qquad \boldsymbol{\varphi} = \frac{number\ of\ opposite\ pairs}{total\ number\ of\ pairs}$$

and in the case of pairing to a more general curve $\sigma_{\mathcal{A}}(u) = \sigma(u) + \sum_{pairs<0}(\lambda_j + \lambda'_j)$. Why?

Idea: negative pairs correspond to the exponents associated with the attraction to $\mathcal{A}$: hence do not count for the computation of $\sigma_{\mathcal{A}}$.

The FR will hold, by the C.H., **but with a slope $\boldsymbol{\varphi} < 1$**:

$$\tau p \boldsymbol{\varphi} \sigma, \quad \text{rather than} \quad \tau p \sigma : \quad \text{in fig. } \boldsymbol{\varphi} \simeq \frac{\mathbf{450}}{\mathbf{490}}$$

If true: this will be a "check of reversibility" in NS_{irr}.

More elaborate checks are being attempted: [8, 16] +

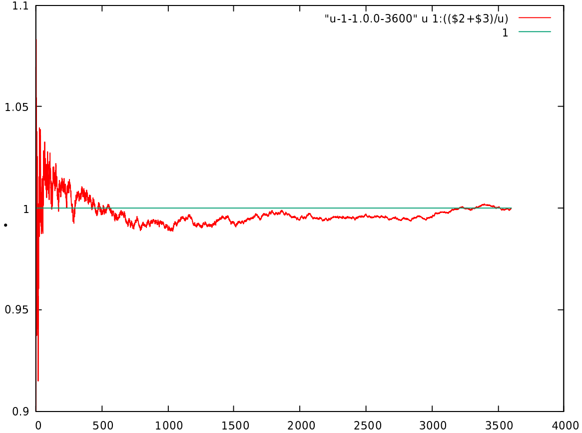
(a) moments of large scale observables rev & irr

(b) local Lyap. exps of matrices different from Jacobian

(c) check of the fluctuation rel., particularly in irrev. cases, (shown above to be accessible already with 960 modes and $R = 2048$): $\Rightarrow$ FR with slope $\boldsymbol{\varphi} < 1$ (Axiom C ?), [15, 17].

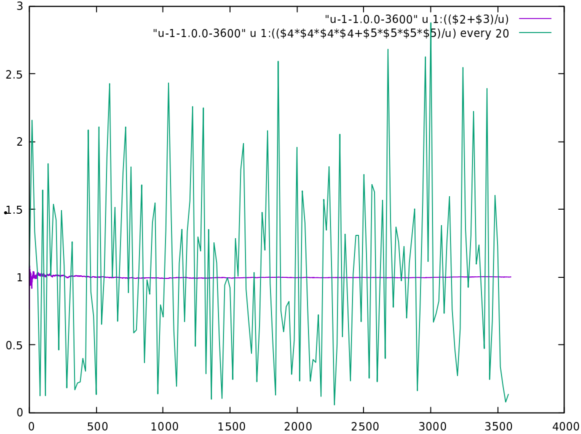
(d) More values of R and N an example with R larger than in the preceding cases yields similar results (not shown).

Example of moments of local observables:



FIGu0-64-191711-10

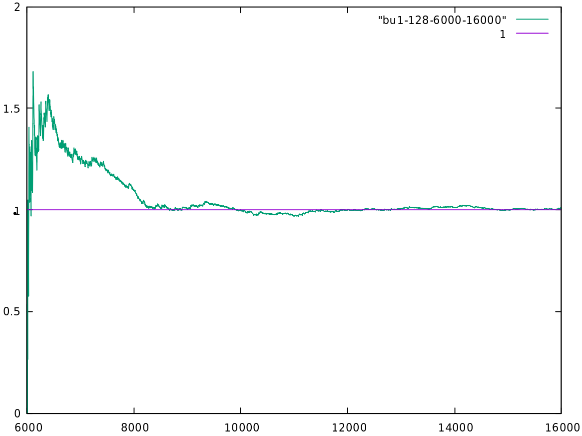
Fig.10: Running averages rev of $(|Re\, u_{11}|^4 + |Im\, u_{11}|^4)/\langle\, |Re\, u_{11}|^4 + |Im\, u_{11}|^4\, \rangle_{irr}$, $R = 2048$, 960 modes. Conjecture yields ratio tending to 1



FIGu1-64-191711-10

Fig.11: Same running averages rev of $(|Re\, u_{11}|^4 + |Im\, u_{11}|^4)/\langle\, |Re\, u_{11}|^4 + |Im\, u_{11}|^4 \,\rangle_{irr}$, for $R = 2048$, and their rev. fluctuations, 960 modes.

Concluding the simulation



FIGA-128

Fig.12: Illustration of the conjecture on a 3968 modes NS: the running average of $R\alpha$ in the reversible NS should tend to 1, according to conjecture.

Finally rigorous estimate of number $\mathcal{N}$ of Lyap. exp. needed so that their sum remains > 0:

$$\lesssim \sqrt{2} A (2\pi)^2 \sqrt{R} \sqrt{R\,En}, A = 0.55..$$

in dimension 2, while at dimension 3 a similar estimate holds but it involves a norm different from the enstrophy. (Ruelle if $d = 3$ and Lieb if $d = 2, 3$, [18, 13].

Applied here it would require $\mathcal{N} \sim 2.10^4$ for NS 2D: not accessible in the simulations presented here but not impossible in principle with available computers and computation methods already available, at least if $D = 2$.

Finally further careful checks are required, particularly since inspiring ideas are, to say the least, **controversial** as shown by the following quote, selected among several, from a well known treatise:

CH is dismissed (by many) with arguments like (1999)

'More recently Gallavotti and Cohen have emphasized the "nice" properties of Anosov systems. Rather than finding realistic Anosov examples they have instead promoted their "Chaotic Hypothesis": if a system behaved "like" a [wildly unphysical but well-understood] time reversible Anosov system there would be simple and appealing consequences, of exactly the kind mentioned above. Whether or not speculations concerning such hypothetical Anosov systems are an aid or a hindrance to understanding seems to be an aesthetic question., [19].

Avoiding to comment on the statement I stress that Statistical Mechanics, from Clausius, Boltzmann and Maxwell has been a simple, surprising, consequence of the **"[wildly unphysical but well-understood]"** periodicity of the collective motions of 10^{19} gas molecules, [20].

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