# Navier-Stokes equation: irreversibility, turbulence and ensembles equivalence

**Question:** can the phenomenological notion of friction be represented in alternative ways?

**Related (?) Q.** is it possible to set up a theory of statistical ensembles, and their equivalence, extending to stationary non-equilibria the ideas behind the canonical and microcanonical ensembles.

**Guide:** *a fundamental symmetry like "time reversal" cannot be "spontaneouly broken"*

*Therefore even the stationary states of dissipative systems ought to be describable via time reversible equations.*

*It will be better to specialize on a paradigmatic example, the NS fluid in a $2\pi$-periodic box, 2/3-D. $R \equiv \frac{1}{\nu}$ be Reynolds #.*

*NS$_{irr}$:* $\dot{u}_\alpha = -(\vec{u}\cdot\boldsymbol{\partial})u_\alpha - \partial_\alpha p + \frac{1}{R}\Delta u_\alpha + F_\alpha, \qquad \partial_\alpha u_\alpha = 0$

*Velocity:* $\vec{u}(x) = \sum_{\vec{k}\neq\vec{0}} u_{\mathbf{k}} \frac{\mathbf{k}^\perp}{|\mathbf{k}|} e^{i\mathbf{k}\cdot\mathbf{x}},$

*NS$_{2,irr}$:* $\dot{u}_{\mathbf{k}} = \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} \frac{(\mathbf{k}_1^\perp\cdot\mathbf{k}_2)(\mathbf{k}_2^2-\mathbf{k}_1^2)}{2|\mathbf{k}_1||\mathbf{k}_2||\mathbf{k}|} u_{\mathbf{k}_1} u_{\mathbf{k}_2} - \nu\mathbf{k}^2 u_{\mathbf{k}} + f_{\mathbf{k}}$

*Although the 2D-NS admit general smooth solution it is convenient to imagine it (aiming at 3D-NS) as truncated at* $|\mathbf{k}| \le N$. *The UV-cut-off $N$ will be fixed for a while.*

*The 2D NS become $4N(N+1)$ ODE's, on phase space $M_N$. (In 3D $O(N^3)$).*

*$Iu_\alpha = -u_\alpha$ does not imply $IS_t = S_{-t}I$, $\Rightarrow$: these are irreversible equations.*

*Let $u$ be an initial state: then $t \to S_t u$ evolves and generates a stationary state on $M_N$ which, aside exceptions collected in a 0-volume in $M_N$, is supposed unique, for simplicity. Let $\mu_R(du)$ be its PDF.*

*Stationary PDFs generalize equilibrium ones: thus collection $\mathcal{E}^c$ of the $\mu_R(du)$ will be called an* **ensemble of nonequil. distrib.** *for $NS_{irr}$.*

*Hence average energy $E_R$, average dissipation $En_R$, (local) Lyapunov spectra $\mathcal{L}_R$ ..., will be defined, e.g.:*

$$E_R = \int_{M_N} \mu_R(du)||u||_2^2, \qquad En_R = \int_{M_N} \mu_R(du)||\mathbf{k}u||_2^2$$

*Consider the new equation, $NS_{rev}$:*

$$\dot{\mathbf{u}}_{\mathbf{k}} = -\sum_{\mathbf{k_1}+\mathbf{k_2}=\mathbf{k}} \frac{(\mathbf{k}_1^{\perp}\cdot\mathbf{k_2})(\mathbf{k_2}\cdot\mathbf{k_1})}{|\mathbf{k_1}||\mathbf{k_2}||\mathbf{k}|}\mathbf{u}_{\mathbf{k_1}}\mathbf{u}_{\mathbf{k_2}} - \alpha(\mathbf{u})\mathbf{k}^2\mathbf{u}_{\mathbf{k}} + F_{\mathbf{k}}$$

*with $\alpha$ s. that $En(u) = ||\mathbf{k}u||_2^2$ is exact constant of motion:*

$$\alpha(u) = \frac{\sum_{\mathbf{k}} \mathbf{k}^2 Re(F_{-\mathbf{k}}u_{\mathbf{k}})}{\sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2} \qquad \text{if } D = 2$$

*The new equation is reversible: $IS_tu = S_{-t}Iu$ (as $\alpha$ is odd).*

*So $\alpha$ is "reversible friction"; (if $D = 3$ slightly different)*

*This can be thought as a "thermostat" acting on the system and it should (?) have same effect as constant friction.*

*The evolution with $NS_{rev}$ generates a family of stationary distributions on phase space: $\mu^{rev}_{En}$ parameterized by the constant value of the dissipation $En = \sum_{\mathbf{k}} |\mathbf{k}|^2 |u_{\mathbf{k}}|^2$. Denote $\mathcal{E}^{rev}$ such collection of stationary PDFs.*

*The $\alpha(u)$ in $NS_{rev}$ will fluctuate strongly if the Reynolds number is large and it will "self-average" to a constant $\nu$ thus "homogenizing" the equation and turning it into the $NS_{irr}$ with friction $\nu$. A first more precise statement:*

*The averages of large scale observables will show the same statistical properties, as $R \to \infty$, in the $NS_{irr}$ and in the $NS_{rev}$ equations under the correspondence*

$$\mu^{irr}_R \longleftrightarrow \mu^{rev}_{En} \quad if \quad \mu^c_R(En(u)) = En$$

*By large scale observables it is simply meant "observables depending on the Fourier's components $u_{\mathbf{k}}$ with $|\mathbf{k}| < K$ with some fixed $K$". And given $K$ and such an observable it should be*

$$\mu^{irr}_R(O) = \mu^{rev}_{En}(O)(1 + o(1/R)) \qquad \textbf{if}$$

$$\mu^{rev}_{En}(\alpha) = \frac{\mathbf{1}}{\mathbf{R}} \qquad \text{or} \qquad \mu^{\mathbf{irr}}_{\mathbf{R}}(||\mathbf{ku}||^{\mathbf{2}}) = \mathbf{En}$$

*Recalls canon.-microcan. equivalence: $\nu = \frac{1}{R}$ plays the role of the canonical temperature ($\frac{1}{\beta}$) and $En$ that of microcanonical energy.*

*Is the limit $R \to \infty$, or strong chaos, the analogue of the thermodynamic limit?*

*The conjecture presented here is* **no** *for equations, like NS, which follow from fundamental microscopic dynamics.*
**Because** *for NS* **much more** *might hold.*

$< 0$ **Examples:** *(not "fundamental")*
*(1) (highly) truncated NS equations ($N < \infty$), [1],*
*(2) NS with Ekman friction, [2, 3],*
*(3) Lorenz96 model, [4],*
*(4) Turbulence shell model, (GOY), [5]*
*where the equivalence is possibly achieved only in the limit of infinite forcing, $R \to \infty$..*

$> 0$ **Examples:** *("fundamental")*
*(1) The NS-equation: which can be derived from first principles. For instance for $NS_{irr}$ (derived by Maxwell from molecular motion, [6]) it is natural to think that there* **should be no condition for strong chaos**.
*The microscopic motion is always strongly chaotic and the chaoticity condition should be always fulfilled even when* **motion appears laminar**.

To pursue this suggestion consider the truncated $NS_{rev/irr}$ equations at momentum $N$: in dimension 2 or 3. Then

The large scale observables, depending on the modes $|\mathbf{k}| < K$, have a the same statistics in corresponding PDFs in $\mathcal{E}^{irr}$ and $\mathcal{E}^{rev}$ in the limit $N \to \infty$ for all $R$ or $En$

The analogy with Equilibrium Stat. Mech. is clear:

(a) The (necessary if $D = 3$) cut-off $N$ plays the role of the finite volume container

(b) the short scale cut-off $K$ restricts attention to local observables

c) the Reynolds number $R$ plays the role of inverse temperature $\beta$ and the dissipation $En$ the role of the microcanonical energy.

**Then**

$$\lim_{N\to\infty} \mu^{rev}_{En}(O) = \lim_{N\to\infty} \mu^{irr}_{R}(O)$$

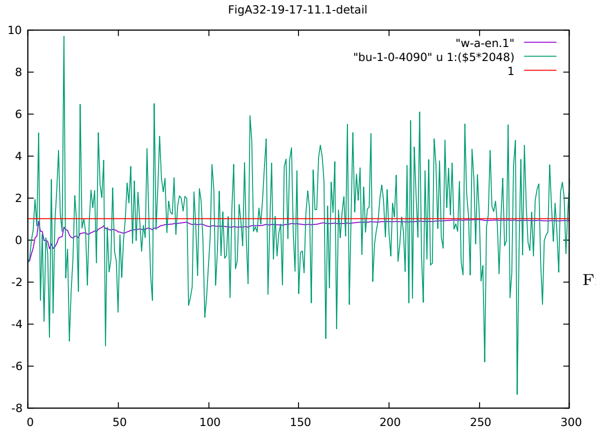
for $O(u)$ depending on $u_{\mathbf{k}}$ with $|\mathbf{k}| < K$ and under the equivalence relation (*i.e.* $\mu^{rev}_{En}(\alpha) = \frac{1}{R}$): of course the larger $K$ the larger $N$ needs to be, just as in equilibrium Stat. Mech.

The above equivalence conjectures suggest measurements on fluids revealing "hidden" reversibility of the motions.

At this point it is convenient to pause and show a few simulations which begin to test the equivalence proposal. Remark the work $W(\mathbf{u}) \stackrel{def}{=} \sum_{\mathbf{k}} \overline{\mathbf{f}}_{\mathbf{k}} \mathbf{u}_{\mathbf{k}}$is **local** and averages as

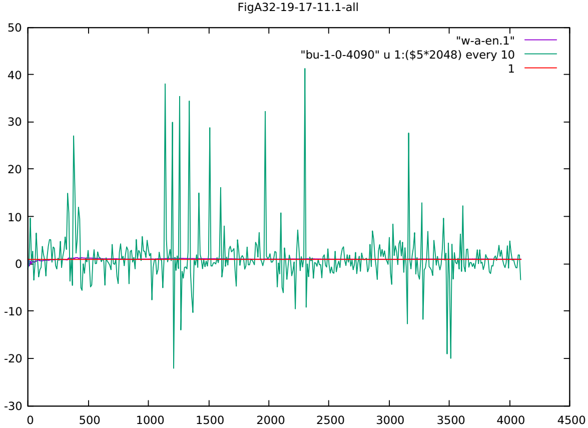
$$W^{irr} \equiv \nu\mu^{irr}_{\nu}(\mathcal{D}), \quad W^{rev} \equiv \mu^{rev}_{En}(\alpha) En$$

**exactly**: therefore implies: $R\langle\, \alpha(\mathbf{u})\,\rangle \equiv 1$, for both Eq.



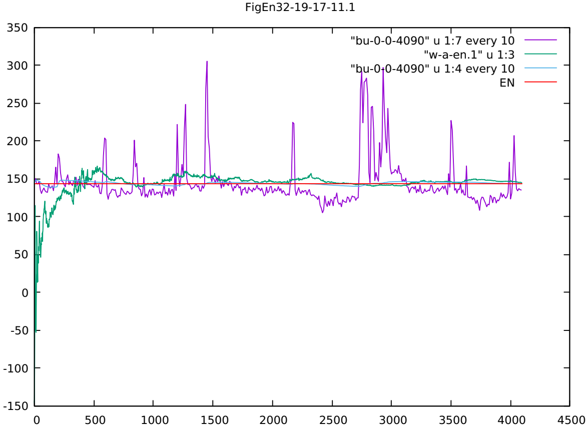

FigA32-19-17-11.1-detail

Fig.1-detail: The running average of the reversible friction $R\alpha(u) \equiv R\frac{2Re(f_{-\mathbf{k}_0} u_{\mathbf{k}_0})\mathbf{k}_0^2}{\sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2}$, superposed to the conjectured value 1 and to the fluctuating values $R\alpha(u)$: Evolution $NS_{rev}$, **R=2048**, 224 modes, Lyap.$\simeq$ 2, x unit $2^{19}$



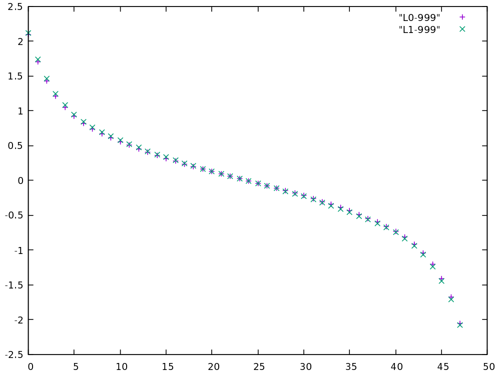

FigA32-19-17-11.1-all

Fig.1: The running average of the reversible friction $R\alpha(u) \equiv R\frac{2Re(f_{-\mathbf{k}_0} u_{\mathbf{k}_0})\mathbf{k}_0^2}{\sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2}$, superposed to the conjectured value 1 and to the fluctuating values $R\alpha(u)$: Evolution $NS_{rev}$, **R=2048**, 224 modes, Lyap.$\simeq$ 2, $x$-axis unit $2^{19}$



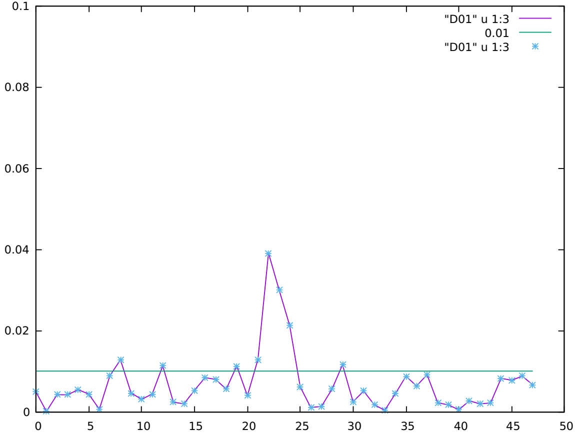

FigEN32-19-17-11.1

Fig.2: Running average of $R\sum_{\mathbf{k}} F_{-\mathbf{k}} u_{\mathbf{k}}|$ (**dark green**) $NS_{rev}$ converges to the average of $\sum_{\mathbf{k}} \mathbf{k}^2 |u_{\mathbf{k}}|^2$ (straight **red** line)
**green** line = running average of $\sum_{\mathbf{k}} \mathbf{k}^2 |u_{\mathbf{k}}|^2$ in $NS_{irr}$
large **fluctuations** are those of $\sum_{\mathbf{k}} |u_{\mathbf{k}}|^2$, $NS_{irr}$: **R=2048**.



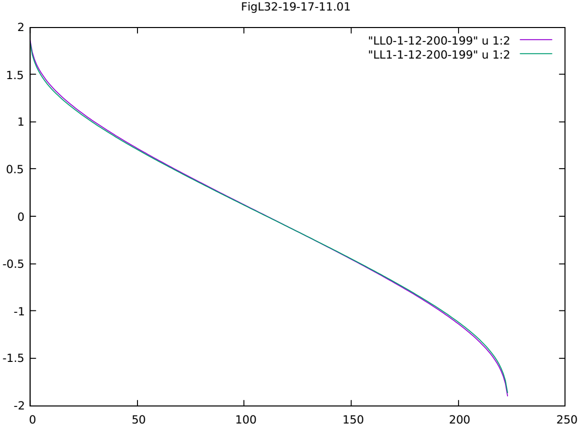

FigL16-15-13-11.01

Fig.3: The (**local**) Lyapunov spectra for 48 modes truncation: reversible and irreversible. And almost pairing, **R=2048**.




FigDiff16-30-15-13-11.01-15

Fig.4: Relative difference betweeen (local) Lyapunov exponents in the previous Fig. R=2048, 48 modes.




FigL32-19-17-11.01

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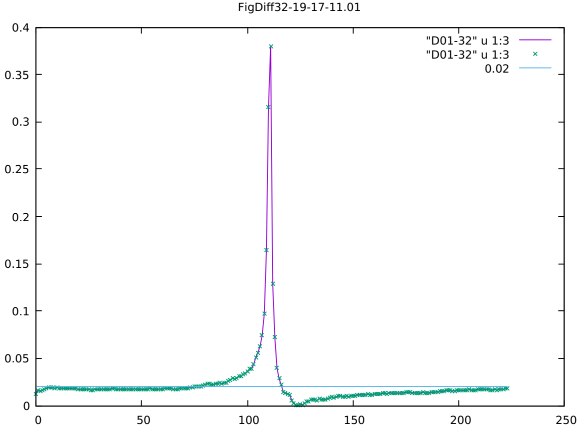
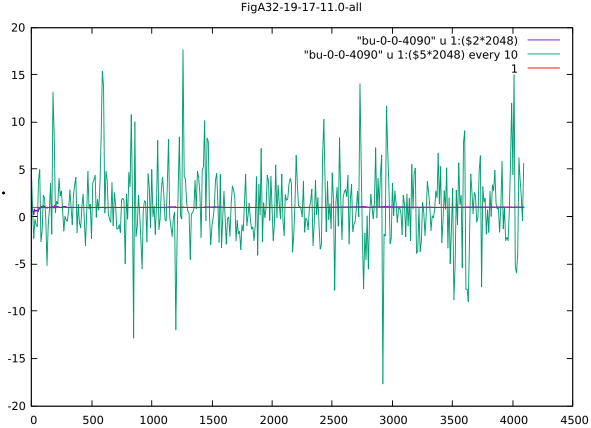
Fig.5: **Local** Lyapunov spectra in a $15 \times 15$ truncation (i.e. for the NS2D with viscosity and reversible viscosity (captions ending respectively in 0 or 1), interpolated by lines, $R = 2048$, 240 modes. are loc. ($2^{13}$ steps) spectra evaluated, every $2^{19}$ int. steps (averaged over 200 steps).




FigDiff32-19-17-11.01

Fig.6: **Relative difference** betweeen (local) Lyapunov exponents in the previous Fig. R=2048, 48 modes. The line is the 2% **level**.

The following Fig.7 (similar to Fig.1 but w. $NS_{rev}$):




FigA32-19-17-11.0-all

Fig.7: The running average of the reversible friction $R\alpha(u)$ as seen by $NS_{irr}$, superposed to the conjectured value 1 and to the fluctuating values $R\alpha(u)$ also in the irreversible $NS_{irr}$. Data correspond to those in Fig.1 above which came from $NS_{rev}$.

Suggests (from the theory of Anoosov systems):
(1) **Test** the "Fluctuation Relation" in the linearized **irreversible** evolution of the Jacobian: if $p = \frac{1}{\tau}\int_0^\tau \frac{\sigma(t)}{\langle\,\sigma\,\rangle} dt$ is finite time average of the **reversible friction** $(\sigma(u) = -\sum_{\mathbf{k}} \partial_{\mathbf{k}}(\dot u_{\mathbf{k}})_{rev})$ then

$$\frac{P_{srb}(p)}{P_{srb}(-p)} = e^{\tau p \langle\,\sigma\,\rangle}\ (\text{as large deviat.as}\tau \to \infty)$$

a "*reversibility test on the irreversible flow*".

(2) **If** FR is respected then a new ensemble $\mathcal{E}^{st}$ can be introduced consisting in the stationary states for the $NS_{st}$

$$\dot u_\alpha = -(\vec u \cdot \boldsymbol{\partial})u_\alpha - \partial_\alpha p + \nu(u)\Delta u_\alpha + F_\alpha, \qquad \partial_\alpha u_\alpha = 0$$

where $\nu(u)$ is a gaussian process uncorrelated in time but with average $\langle\,\nu\,\rangle = \frac{1}{R}$ and PDF respecting the FR (*i.e.* dispersion equal to the average)

Anosov systems play the role, in chaotic dynamics, of the harmonic oscillators in ordered dynamics. They are the paradigm of Chaos.

This idea rests on the work of **Sinai** (on Anosov sys.), **Ruelle, Bowen** (on Axioms A sys.),[7, 8, 9]

Accent on Anosov sys. has led to the

**Chaotic hypothesis:** *A chaotic evolution takes place on a smooth surface $\mathcal{A}$, "attracting surface", contained in phase space, and on $\mathcal{A}$ the maps $S$ (or the flow $S_t$) is an Anosov map (or flow).*

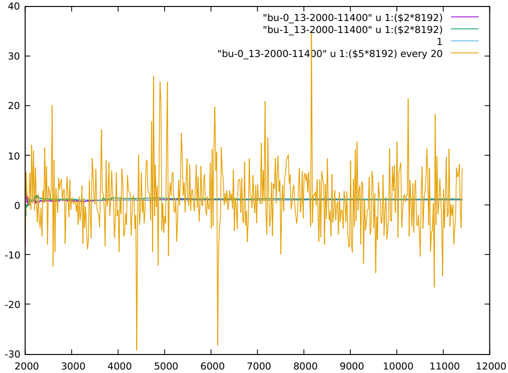
A strict, general, heuristic, interpretation of original ideas on turbulence phenomena, [9], see [10, endnote 18], [11, 12], [13].

More elaborate tests are under way:

(a) moments of large scale observables rev & irrev

(b) study (local) Lyapunov exponents of other matrices instead of the Jacobian

(c) there is evidence that already with 224 modes the dimension of the attracting surface is lower than the phase space dimension: $\Rightarrow$ Fluct. Rel. with slope $< 1$ (Axiom C ?), [12, 11].
Other matrices can have exponents much larger hence (local) L. exp. may be easier to compute. Only preliminary results are available.




FigA.0-13-2000-11400-13

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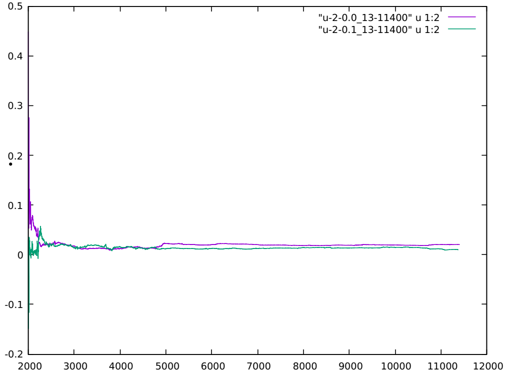
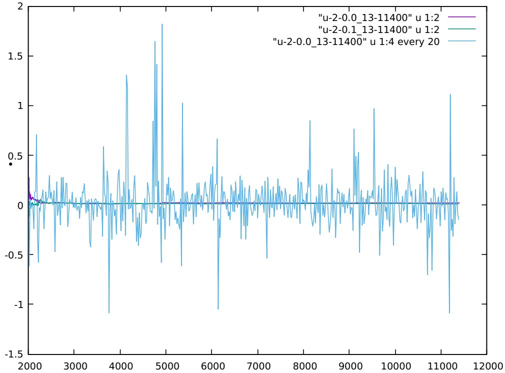
Fig.8: Higher $R = 8192$, 224 modes: running averages of $R\alpha(u)$ for $NS_{irr}$ & $NS_{rev}$, (predicted 1) and fluctuations for the $NS_{irr}$. Time recorded every $4\lambda_{max}^{-1}$.



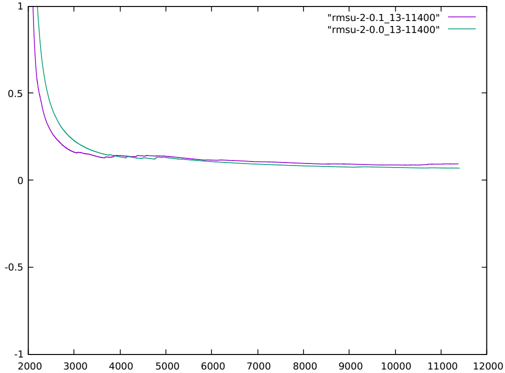

Figu20-0/1-19-17-13

Fig.9: Running averages rev/irr of the $|u_{20}|^2$ component, $R = 8192$, 224 modes.




Figu20-0/1-19-17-13

Fig.10: Same running averages rev/irr of the $|u_{20}|^2$ component, $R = 8192$, plus the fluctuations in the irr case, 224 modes.




FIGrmsu20-0/1-19-17-13

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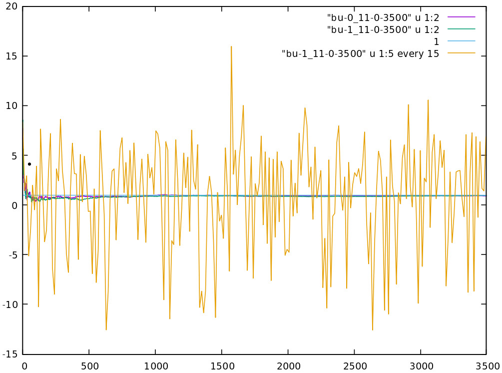
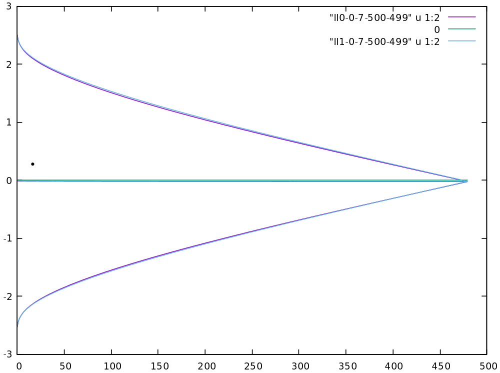
Fig.11: RMS for the above $|u_{20}|^2$ rev/irr, $R = 8192$, 224 modes



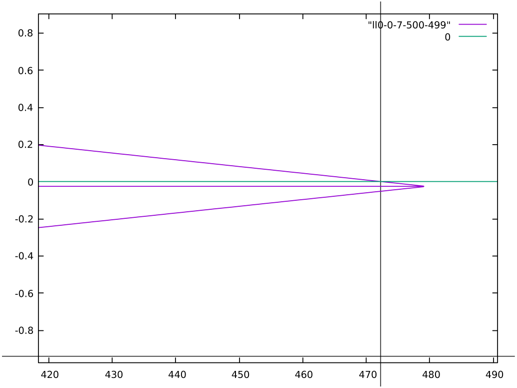

FIGA-64-19-17-11

Fig.12: $R = 2048$, 960 modes, running averages of $\alpha(\mathbf{u})$ and its fluctuations in the reversible evolution (in the irreversible evolution similar fluctuations).




FIGll-64-19-17-11

Fig.13: $R = 2048$, 960 modes, **Local** exponents ordered by decreasing values $\lambda_k$, $0 \le k < d/2$, and increasing $\lambda_{d-k}, 0 \le k < d/2$ and the lines $\frac{1}{2}(\lambda_k + \lambda_{d-1-k})$ amd $\equiv 0$.




FIGll-detail64-19-17-11

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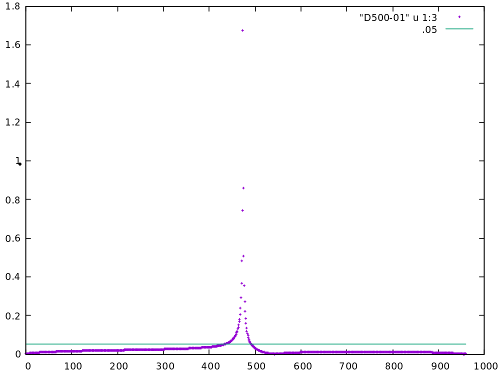
Fig.14: Detail of the previous figure showing the irreversible exponents (only) and the line $\equiv 0$ which illustrates the dimensional loss of $\frac{472}{490}$. See Ruelle: [14, Eq.(1.7)]. $R = 2048$, 960 modes.




FIGdiff64-19-17-11

Fig.15: Relative difference $\frac{|\lambda_k^{rev} - \lambda_k^{irr}|}{\max(|\lambda_k^{rev}|, |\lambda_k^{irr}|)}$ between reversible and irreversible local exponents in Fig.13. The line is the 5% **level**.

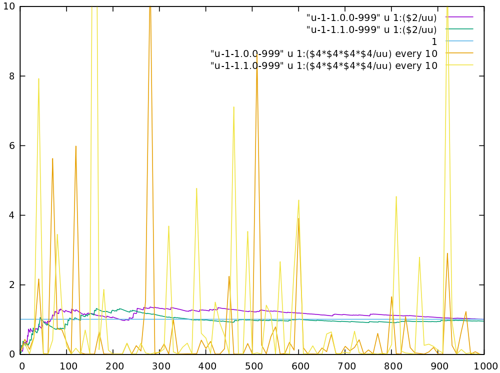



FIG16-u4-15-13-11.01

Fig.16: graph of $u_{1,1}^4 / \langle\, u_{1,1}^4 \,\rangle$, reversible and irreversible evolutions with running averages and fluctuations.
$R = 2048$ and 48 modes. **Typical** check of the conjecture.

Finally one can estimate the number of descreasing **local** exponents such that $\sum_i \lambda_i \le 0$ (KY dimension?)

$$N \le \sqrt{2}A(2\pi)^2\sqrt{R}\sqrt{RD}, A = 0.55..$$

which holds **rigorously** in dimension 2, while in dimension 3 a similar estimate holds but it is expressed in terms of a different norm than $D$. The estimate, due to Ruelle if $d = 3$ and Lieb if $d = 2$, [15, 16], gives here (since $RD = 1$, according to the conjecture and the simulations): $N \simeq 1350$ which cannot be tested because in in our simulations the number of modes is $\le 960$, yet it is compatible and it might be tested soon.

CH is dismissed (by many) with arguments like (1999)

'More recently Gallavotti and Cohen have emphasized the "nice" properties of Anosov systems. Rather than finding realistic Anosov examples they have instead promoted their "Chaotic Hypothesis": if a system behaved "like" a [wildly unphysical but well-understood] time reversible Anosov system there would be simple and appealing consequences, of exactly the kind mentioned above. Whether or not speculations concerning such hypothetical Anosov systems are an aid or a hindrance to understanding seems to be an aesthetic question., [17].

While giving up evaluating the statement I stress that Statistical Mechanics, after Clausius, Boltzmann and Maxwell was a simple and appealing consequence of the **"[wildly unphysical but well-understood]"** periodicity of collective motions the $10^{19}$ atoms in a gas, [18].

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