Statistical ensembles for Navier-Stokes equation

Statistical properties of an Equilibrium state are obtained by several different probability distributions, *e.g.* canonical or microcanonical: which attribute the same average to physically interesting obervables. **Reminder**:

The probability distr. describing a system with ρV particles in volume V can be collected in families $\mathcal{E}^{mc}, \mathcal{E}^{c}, \ldots$ whose elements are parameterized by parameter E or, resp., β.

1) observables of interest are local observables $O \in \mathcal{O}_{loc}$: $O(\mathbf{p}, \mathbf{q})$ depending on $\mathbf{p}, \mathbf{q}$ only through coordinates of particles $q_i \in \mathbf{q}$ with $q_i \in \Lambda$ where Λ is a volume $\ll V$

2) the probability distribution $\mu_\beta^V \in \mathcal{E}^c$ and $\widetilde{\mu}_E^V \in \mathcal{E}^{mc}$ are **correspondent** if β, E are s.t.

$$\mu_\beta^V(H_V(\mathbf{p}, \mathbf{q})) = E$$

Then

$$\lim_{V\to\infty} \mu_\beta^V(O) = \lim_{V\to\infty} \widetilde{\mu}_E^V(O)$$

and μ's are *equivalent* in the thermodynamic limit.

In case of phase transitions extra labels $\gamma, \widetilde{\gamma}$ are added to identify the extremal distributions and it is possible to establish a correspondence between the extra labels $\gamma \longleftrightarrow \widetilde{\gamma}$ so that the equivalence can be equally formulated.

Is it possible a similar description of the stationary states of nonequilibrium systems?

Think of a system whose evolution is described by an evolution eq. of u on a "phase space" M depending on a parameter R:

$$\dot{u} = f_R(u)$$

Typically eq. will be difficult and even existence-1-qness will be open problems.

For instance consider a system of infinitely many hard spheres of given density or an incompressible 3D NS fluid with periodic b.c.

Therefore the eq. will have to be regularized in $f_R^V(u)$ where V is a regularization parameter.

E.g. in stat. mechanics V is typically the container size: and the problem becomes finding the observables whose averages have a limit as $V \to \infty$. They exist and are $O(u)$ which only depend on the points of u in a region $K \ll V$, local observables.

For the NS equation the regularization parameter could be a "UV cut-off" N. And it is natural to consider as observables whose average admit a limit as $N \to \infty$ the $O(u)$ which only depend on the Fourirer's components $\mathbf{k}$ of u ith $|\mathbf{k}| < K \ll N$.

Once the class of observables is restricted it is to be expected (?) that several equations of motion could describe the stationary states of the same system.

E.g. the h.c. system can be described by the Hamilton eq.s but also by the isothermal equations

$$\dot{\mathbf{q}} = \mathbf{p}, \qquad \dot{\mathbf{p}} = -\boldsymbol{\partial}_{\mathbf{q}} V(\mathbf{q}) - \alpha(\mathbf{p}, \mathbf{q})\mathbf{p}$$

where $\alpha(\mathbf{p}, \mathbf{q})$ is a multiplier which imposes $T(\mathbf{p}) = const.$

The stationary states of the two equations will be parameterized by the energy E or by the kinetic energy T; stationary states will be resp. $\delta(H(\mathbf{p}, \mathbf{q}) - E)d\mathbf{p}d\mathbf{q}$ or

$$e^{-\beta_0 V(\mathbf{q})}\delta(T(\mathbf{p}) - N\beta^{-1})d\mathbf{p}d\mathbf{q}, \qquad \beta_0 = \beta(1 - \frac{1}{3N})$$

Interesting cases arise when the system is described by equations which obey a symmetry but they are phenomenologically described by non symmetric equations (cases of spontaneously broken symmetry).

Consider, as a typical case, the Navier-Stokes equation: in the case of the above incompressible fluid they can be regarded as Euler equations subject to a thermostat absorbing the heat due to the viscosity: which turns the equations into time-reversal breaking ones.

A paradigmatic case is a fluid in a periodic container 2/3-Dim., incompressible, **at fixed forcing** F (smooth, $\|F\|_2 = 1$) and kept at const. temp. by a thermostat. to dissipate heat via the force due to viscosity $\boldsymbol{\nu} = \frac{\mathbf{1}}{\mathbf{R}}$ (consistently with incompressibility).

NS_{irr}: $\dot u_\alpha = -(\vec u\cdot\boldsymbol{\partial})u_\alpha - \partial_\alpha p + \frac{1}{R}\Delta u_\alpha + F_\alpha, \qquad \partial_\alpha u_\alpha = 0$

Velocity: $\vec u(x) = \sum_{\vec k\neq\vec 0} u_{\mathbf k}\frac{i\mathbf{k}^\perp}{|\mathbf k|}e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \overline{u}_{\mathbf k} = u_{-\mathbf k}$ (NS-2D)

$NS_{2,irr}$: $\dot u_{\mathbf k} = \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} \frac{(\mathbf{k}_1^\perp\cdot\mathbf{k}_2)(\mathbf{k}_2^2-\mathbf{k}_1^2)}{2|\mathbf{k}_1||\mathbf{k}_2||\mathbf{k}|} u_{\mathbf{k}_1}u_{\mathbf{k}_2} - \nu\mathbf{k}^2 u_{\mathbf k} + f_{\mathbf k}$

Immagine to truncate eq. supposing $|\mathbf{k}_j|\le V$. Cut-off UV, V, is temporarily fixed (**BUT interest is on $V\to\infty$**).

NS 2D becomes an ODE in a phase space M_V with $4V(V+1)$ dimen. (In 3D $O(8V^3)$). Exist. & 1-ness trivial at $D=2,3$.

Remark that the map $Iu_\alpha=-u_\alpha$ implies $IS_t\neq S_{-t}I$, $\Rightarrow$: irreversibility.

Given init. data u, evolution $t\to S_t u$ generates a steady state (*i.e.* a probability distr.) $\mu_R^{irr,V}$ on M_V.
Unique aside a volume 0 of u's, for simplicity

Likely not so at small R: "NS gauge symmetry" exists..**??** [1, 2, 3]. As R varies the steady distr. $\mu_R^{irr,V}(du)$ form a collection $\mathcal{E}^{irr,V}$: to be named

the statistical ensemble of stationary nonequilibrium distrib. for NS_{irr}.

And average energy E_R, average dissipation En_R, Lyapunov spectra (local and global) ... will be defined, *e.g.*:

$$E_R = \int_{M_V} \mu_R^{irr,V}(du)||u||_2^2, \qquad En_R = \int_{M_V} \mu_R^{irr,V}(du)||\mathbf{k}u||_2^2$$

Consider new equation, NS_{rev}:

$$\dot{\mathbf{u}}_{\mathbf{k}} = \sum_{\mathbf{k_1}+\mathbf{k_2}=\mathbf{k}} \frac{(\mathbf{k_1^\perp}\cdot\mathbf{k_2})(\mathbf{k_2^2}-\mathbf{k_1^2})}{\mathbf{2}|\mathbf{k_1}||\mathbf{k_2}||\mathbf{k}|}\mathbf{u}_{\mathbf{k_1}}\mathbf{u}_{\mathbf{k_2}} - \alpha(\mathbf{u})\mathbf{k^2}\mathbf{u}_{\mathbf{k}} + f_{\mathbf{k}}$$

with α such t. $En(u) = ||\mathbf{k}u||_2^2$ is **exact const of motion**:

$$\alpha(u) = \frac{\sum_{\mathbf{k}} \mathbf{k}^2 F_{-\mathbf{k}} u_{\mathbf{k}}}{\sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2} \qquad e.g.\ \ D = 2$$

The new equation keeps $\nu \sum_{\mathbf{k}} |\mathbf{k}|^2 |\mathbf{u_k}|^2 = \nu \cdot enstrophy$ exactly constant

New eq. is reversible: $IS_t u = S_{-t} I u$ (as α is odd).
α is "a reversible viscosity"; (if $D = 3$ α is ∼different)

Can be considered as model of "thermostat" acting on the fluid and should (?) have **same effect** of constant friction.

Evolution NS_{rev} generates a family of steady states $\mathcal{E}^{rev,V}$ on M_V: $\mu_{En}^{rev,V}$ parameterized by the constant value of **enstrophy** $En = \sum_{\mathbf{k}} |\mathbf{k}|^2 |u_{\mathbf{k}}|^2$.

$\alpha(u)$ in NS_{rev} **will wildly fluctuate** at large R ($i.e.$ small viscosity ν) thus "**self averaging**" to a const. value ν "homogenizing" the eq. into NS_{irr} with viscosity ν.

Of course we could impose a multiplier $\alpha'(u) = \frac{\sum_{\mathbf{k}} f_{\mathbf{k}} \overline{u}_{\mathbf{k}}}{\sum_{\mathbf{k}} |\mathbf{k}|^2 |u_{\mathbf{k}}|^2}$ which fixes energy $E = \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2$ and obtain **diff. rev. eq.**

The equivalence mechanism is suggested by analogy with Stat. Mech.

(1) analog of "local observables": functions $O(u)$ which depend only on $u_{\mathbf{k}}$ with $|\mathbf{k}| < K$. "Locality in momentum"
(2) analog of "Volume": just the cut-off N confining the $\mathbf{k}$
(3) analog of the "state parameter": the viscosity $\nu = \frac{1}{R}$ (irrev. case) or the enstrophy En (rev. case) (or energy E).

Equivalence should be obtained at $N = \infty$ corresponding to the Thermodynamic limit $V \to \infty$.

The averages of large scale observables will tend to the same values as $R \to \infty$ for $\mu_R^{irr,V} \in \mathcal{E}^{irr,V}$ of NS_{irr} and for $\mu_{En}^{rev,V} \in \mathcal{E}^{rev,V}$ provided, $\mathcal{D}(\mathbf{u}) \stackrel{def}{=} \sum_{\mathbf{k}} \mathbf{k}^2 |\mathbf{u}_{\mathbf{k}}|^2$ is s.t.

$$\mu_R^{irr,V}(\mathcal{D}) = En, \qquad \text{or} \qquad \mu_{En}^{rev,V}(\alpha) = \frac{1}{R}$$

Remark that multiplying the NS eq. by $\overline{u}_{\mathbf{k}}$ and sum on $\mathbf{k}$:

$$\frac{1}{2}\frac{d}{dt}\sum_{\mathbf{k}}|u_{\mathbf{k}}|^2=-\gamma\mathcal{D}(\mathbf{u})+W(\mathbf{u}),\quad \gamma=\nu\ or\ \alpha(\mathbf{u})$$

here $\mathcal{D}(\mathbf{u})=\sum_{\mathbf{k}}\mathbf{k}^2|\mathbf{u}_{\mathbf{k}}|^2=$ enstrophy and
$W=\sum_{\mathbf{k}}\mathbf{f}_{\mathbf{k}}\mathbf{u}_{-\mathbf{k}}=$ work per unit time of the external force.
Hence time averaging

$$\frac{1}{R}\mu_R^{irr,V}(\mathcal{D})=\mu_R^{irr,V}(W),\qquad \mu_{En}^{rev,V}(\alpha)En=\mu_{En}^{rev,V}(W)$$

But W is local (as $\mathbf{f}$ is such) and, if the conjecture holds, has equal average under the equivalence condition: hence $\mu_R^{irr,V}(\mathcal{D})=En$ **implies** the relation

$$\lim_{R\to\infty}R\mu_{En}^{rev,V}(\alpha)=1$$

This becomes a first rather stringent test of the conjecture.

Since the equivalence rests on the rapid fluctuations of $\alpha(u)$ a second idea is that if N is kept finite then it could be, more generally if O is a large scale observable it should be:

$$\mu_R^{irr,V}(O) = \mu_{En}^{rev,V}(O)(1+o(1/R)) \qquad \textbf{if} \qquad \mu_R^{irr,V}(\mathcal{D}) = En$$

So a different idea arises. In many phenomenological and dissipative equations of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \nu\mathbf{x} + \mathbf{g}$ the parameter ν can be replaced by $\alpha(\mathbf{x})$ so that $\mathbf{x}^2$=xonst. If for $\nu = 0, \mathbf{g} = \vec{0}$ the motion is strongly chaotic then

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \nu\mathbf{x} + \mathbf{g},$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \alpha(\mathbf{x})\mathbf{x} + \mathbf{g}, \qquad \alpha(\mathbf{x}) = \frac{\mathbf{g}\cdot\mathbf{x}}{\mathbf{x}^2}$$

Equivalence if $\nu \to 0$ between stationary μ_ν^{irr} and μ_E^{rev} if

$$\mu_\nu^{irr}(\alpha) = E$$

What is special to NS to conj. that $R \to \infty$ is **not** needed?

It is its being a scaling limit of a microscopic equation whose evolution is certainly chaotic and reversible.

Therefore NS is **different** from the many phenomenological and dissipative equations which are not directly related to fundamental equations.

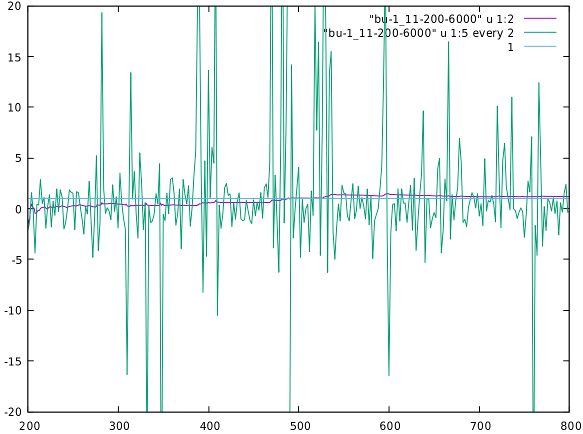
For the latter cases strong chaos is necessary if a friction parameter is changed into a fluctuating quantity.
There are many examples of phenomenological equations

(1) (highly) truncated NS equations ($V < \infty$ fixed), [4],
(2) NS with Ekman friction ($-\nu\vec{u}$ instead of $\nu\Delta\vec{u}$), [5, 6],
(3) Lorenz96 model, [7],
(4) Shell model of turbulence, (GOY), [8]

in such equations $R \to \infty$ is necessary: and, for each of them, it has been tested in few cases.

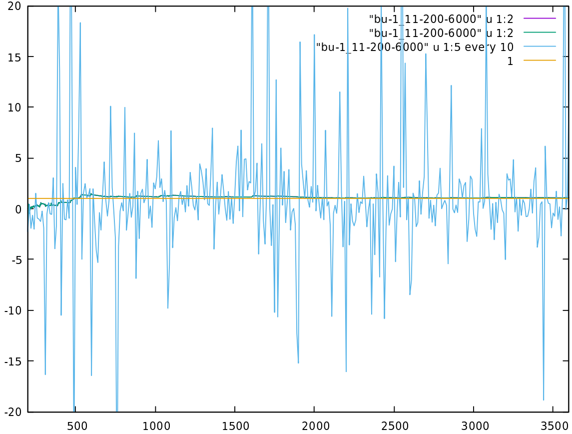
But it will be useful to pause to illustrate a few prelimnary simulations and checks.

Unfortunately the simulations are in dimension 2 ($D = 3$ is at the moment beyond the available (to me) computational tools) although present day available NS codes should be perfectly capable to perform detailed checks in rapid time.



FigA32-19-17-11.1-detail

Fig.1-dettaglio: Running average of reversible friction $R\alpha(u) \equiv R\frac{2Re(f_{-\mathbf{k}_0} u_{\mathbf{k}_0})\mathbf{k}_0^2}{\sum_{\mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}|^2}$, superposed to conjectured 1 and to the fluctuating values of $R\alpha(u)$. Initial transient is clear. Evol.: NS_{rev}, **R=2048**, 224 modes, Lyap. $\simeq 2$, x-unit $= 2^{19}$



FigA32-19-17-11.1-all

Fig.1: As previous fig. but time 8 times longer: data reported "every 10", **or** black.

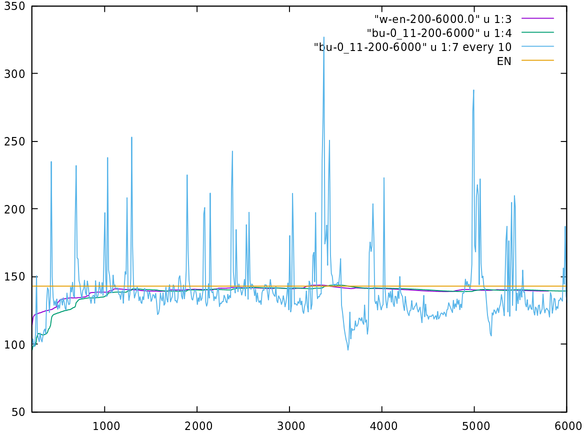


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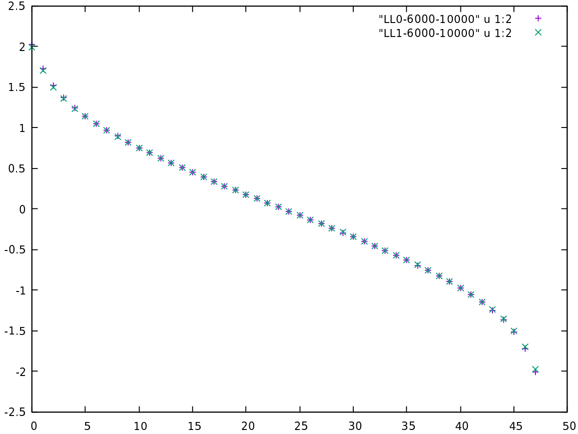
Fig.2: NS_{irr}: **Running** average of the work $R\sum_{\mathbf{k}} F_{-\mathbf{k}} u_{\mathbf{k}}|$ (**violet**) in NS_{rev}; and **convergence** to average enstrophy En (**orange** straight line),

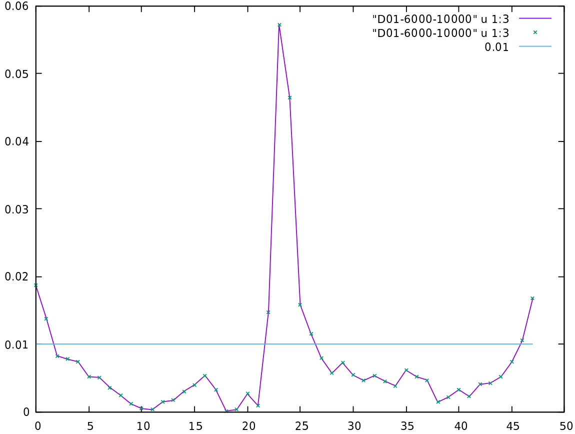
blue is running average of enstrophy $\sum_{\mathbf{k}} \mathbf{k}^2 |u_{\mathbf{k}}|^2$ in NS_{irr}, enstrophy **fluctuations** violet in NS_{irr}: **R=2048**.



FigL16-19-17-11.01

Fig.3: Spectrum (**local**) Lyapunov V=48 modes reversible & irreversible superposed; **R=2048**.

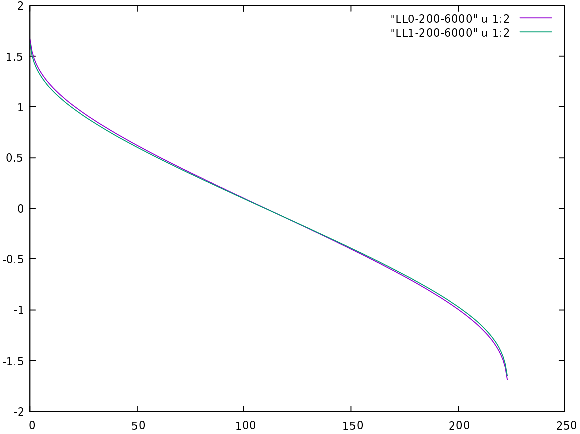
The difference can be made visible as:



FigDiff16-191711-01

Fig.4: **Relative Difference** of (local) Lyap. exponents in Fig. preced. R=2048, 48 modes.

Graph of $\frac{|\lambda_k^{rev}-\lambda_k^{irr}|}{\max(|\lambda_k^{rev}|,\lambda_k^{irr}|)}$; Level line marks 1%.



FigL32-19-17-11.01

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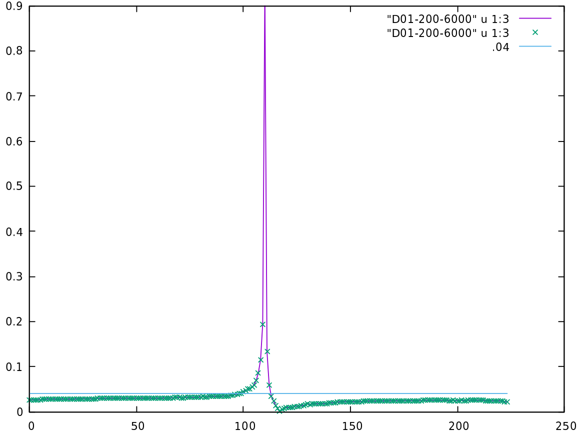
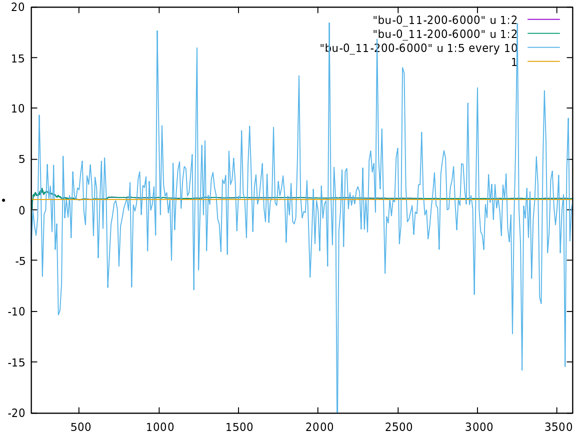
Fig.5: More Lyapunov spectrume in 15×15 modes (i.e. for NS2D rever. & irrev. $R = 2048$, 240 modes on 2^{13} steps. Spectra evalued every 2^{19} integr. steps. (and averaged over 200 samples).



FigDiff32-19-17-11.01

Fig.6: **Relative difference** of the (local) Lyapunov exp. of the preceding fig. 240 modes. The line is the 4% **level**.

The following Fig.7 (similar to Fig.1 but w. NS_{irr}):



FigA32-19-17-11.0-all

Fig.7: As Fig.1 but running average of reversible friction $R\alpha(\mathbf{u})$ regarded as observ. in NS_{irr}, superposed ro value 1 and to fluctuating values of $R\alpha(\mathbf{u})$. An extension of conjecture since $\alpha(\mathbf{u})$ is not local.

The figure suggests (from the theory of Anosov systems):
(1) **Check** the "Fluctuation Relation" in the **irreversible** evollution: for the divergence (trace of the Jacobian) $\boldsymbol{\sigma}(u) = -\sum_{\mathbf{k}} \partial_{u_{\mathbf{k}}} (\dot{u}_{\mathbf{k}})_{rev}$: let p (time τ average of $\frac{\sigma}{\langle \sigma \rangle}$)

$$p \stackrel{def}{=} \frac{1}{\tau} \int_0^\tau \frac{\boldsymbol{\sigma}(\mathbf{u}(t))}{\langle \boldsymbol{\sigma} \rangle_{irr}} dt,$$

then a theorem for Anosov systems:

$$\frac{P_{srb}(p)}{P_{srb}(-p)} = e^{\tau\, \mathbf{1}\, \mathbf{p} \langle \boldsymbol{\sigma} \rangle_{\mathbf{irr}}} \text{ (sense of large deviat. as } \tau \to \infty)$$

it is a "*reversibility test on the irreversible flow*"

Anosov systems play the role, in chaotic dynamics that harmocic oscillators cover for ordered motions.
They are a paradigm of chaos.

The idea is based on **Sinai** (for Anosov syst.), **Ruelle, Bowen** (for Axioms A syst.),[9, 10, 11]

Attention on Anosov syst. leads to:

Chaotic hypothesis: *An empirically chaotic evolution takes eventually place on a smooth surface $\mathcal{A}$, "attracting surface" in phase space and, on $\mathcal{A}$, the evolution (map S or flow S_t) is a Anosov syst.*

It is a strict and general heuristic interpretation of the original ideas on turbulence phenomena, [11], see [12, endnote 18], [13, 14], [15].

BUT: various are the obstacles to its applicability and resolving them leads to new interesting problems.

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