Chaos theory for mechanics and fluids: ergodicity, modern views & controversies

Newtonian physics: atoms interact via conservative forces.

Q. Can atomic hypothesis explain thermodynamics?

Boltzmann and Maxwell: yes if "ergodic hypothesis":

$6N$ coordinates $\vec{x} = (\vec{p}, \vec{q})$ of N isolated atoms on the surfaces $\Sigma_E\{H(\vec{q}) = E\}$ of energy E describe $t \to \vec{x}(t) \equiv \vec{S}_t \vec{x}$ a trajectory which

(1) is dense on Σ_E and
(2) fraction of time spent in a prefixed region Δ is proportional to volume: as $T \to \infty$

$$\mu(\Delta) = \frac{1}{T} \textit{time in } \Delta = \frac{\int_\Delta d^{3N}\vec{p}\, d^{3N}\vec{q}\, \delta(H(\vec{p},\vec{q}) - E)}{\text{surface } \Sigma_E},$$

B. & M. wrote the trajectory visits all points, and deduced that the invariance of $\mu(\Delta)$ implied proportionality to the volume, apparently thinking that the Liouville volume was the only invariant volume, [1, 2].

To summarize:

(1) motion makes a inital datum evolve visiting densely phase space

(2) state is an invariant probability distribution μ on phase space

(3) μ admits a density F with respect to phase space volume

(4) invariance implies F=constant of motion, > 0 on phase space
(no other const. could exist, **generically**...)

Immediately it was stated that hypothesis could not explain "why approach to equilibrium via a reversible evolution. And a visit to **all** phase space points was pointed out *not possible* mathematically and criticism remained alive (until now).

B. & M. **intended ergodicity for "generic" systems** not for all ("of course" ?): literal "second hand" interpretation of their ideas easily led to persisting inconsistence proofs or to expressions of doubt.

This has been the first instance with genericity coming into play in a fundamental way.

Yet ergodicity was a revolutionary idea, and led to statistical mechanics. BUT mathematical examples of ergodic systems appeared much later, aside from the quasi periodic motions, too simple for the skeptics.

Perhaps lack of simple examples has been a main problem for understanding of Ergodic Hypothesis: until '970s we were still trained to understand motion via integrable systems, *i.e.* essentially the harmonic oscillators.

In 1868 B. [3] gave an example (never even quoted before [4, 5]): which however strictly speaking is (probably) incorrect but it illustrates clearly his idea that ergodicity **is a generic property** (for commented description: [5])

Only in the '900's first examples became available: and only in the '960s they begun to influence Statistical Mechanics.

The **"harmonic oscillator"** role was played by the **"Anosov systems"** which should be considered the paradigm of chaotic motion.

The main property is that "any" two close inital data, no matter how close, will separate exponentially in time future or past: **"hyperbolicity"**.

In spite of that such systems are mathematically extremely simple.

I am not going to replace the ergodic hypothesis with the assumption the motion is hyperbolic. It would be too simple and wrong. I want to try to illustrate aspects of the novelties introduced, mainly by Ruelle, and their relevance for Physics.

The modern viepoint is far more ambitious.

It starts with restricting attention to few observables: we do not want a theory of the frequency of visit of a tagged O_2 molecule to $1mm^3$ located somewhere in a closed room. Of course the Ergodic Hypothesis would tell us, but..

For instance in SM we look 'mainly' at "local observables", which depend only on the configuration of particles located in a region Δ small compared to the container size. And we want to restrict attention to "generic systems", possibly even with few particles.

Not as special as Sun+Jupiter or as a pendulum but including a rigid body subject to a constant torque or a Navier-Stokes fluid subject to a constant stirring force.

However genericity is not sufficient: ergodicity is not a property of the equations of motions. It also depends on how the volume $\mu(\Delta)$ is measured: soon is realized that non trivial equations possess ∞-many invariant "volumes".

Hence the argument that Liouville volume is preferred being invariant **cannot be convincing** (also because of the dramatic shape changes undergone by the volumes).

Certainly studying any system one has to prepare the initial conditions: it is illusory to think that they can be given with precision. We can only fix a (reproducible) protocol to be followed to prepare them.

Their data will be determined up to uncontrollable small errors. The two key assumptions for a theory:

I) the protocols can be supposed such that phase space coordinates of the system are determined up to errors randomly distributed with a density over phase space

This is an apparently obvious starting: we are NOT saying that we know the random distribtion of data. We are saying that it exists, depends on the protocol but we have no way to know it. It can (**should**) be claimed that this assumption is a law of nature.
The law has the greatest physical relevance: combined with the second assumption

II) Generically evolutions are hyperbolic (Ruelle) (so you can neglect, in this context, the existence of clocks,...)

This second assumption, once agreed that hyperbolic systems are very simple and accepting I), is essential because of the properties of hyperbolic systems.

(Anosov,Sinai,Ruelle,Bowen, [6]) show for such systems if I) and II) hold then the system (subject constant external forces, 0 or not) will evolve towards a unique stationary state *independent on the unknown (but existent) initial distribution of the data*, or possibly towards a finite number of possible stationary states ("phase transitions").

Applied to, say, a gas in a perfect box explains why it evolves to be described by microcanonical ensemble: **it explains why the Liouville volume is privileged**: not because an evolving phase space volume keeps the same volume. But because the **motion of the gas is chaotic.**

Furthermore the two assumptions of Ruelle have the great merit that they **unify equilibrium and non equilibrium**: in all cases in which motions are chaotic and generated by a given protocol there is a unique stationary probability distribution describing the properties of selected observables (or a finite number, at phase transitions).

In Physics the probability distributions are often called "ensembles" (not to be confused with the often used definition of ensemble as a collection of identical copies of the same systems): different models of the same phenomenon should uniquely lead to the same averages of the selected observables (as well known in equilibrium).

Therefore it is natural that the same phenomena could be described by several equivalent ensembles, once attention is confined to a suitable class of observables. This is **new in nonequilibrium**, leading into totally new territory, This is observed in many experiments and simulations.

What is difficult to understand it the total closure to the basic new ideas by physicists who initiated and continue to develop through simulations a phenomenology that confirms I,II in the study of stationary non-equilibrium phenomena.

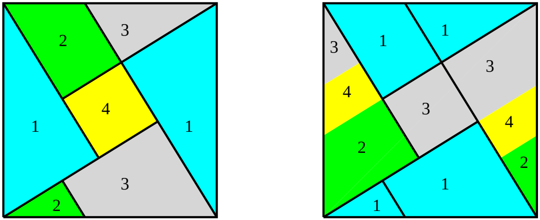
Some of them do not hesitate to shoot inappropriate comments, [7, p.344]: an excerpt:

... has discussed the possibility that the useful properties exhibited by certain oversimplified and quite rare dynamical systems, termed "Anosov systems", have counterparts in the more usual thermostatted systems studied with nonequilibrium simulation methods. Anosov systems are oversimplifications, like square clouds or spherical chickens...

They are unaware that while the map of the square $[0,1]^2$

$$S : \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ x+y \end{pmatrix}$$

is the simplest example of Anosov system



there are

a) concrete mechanical contrivances built with screws and joints which move as Anosov systems, [8]

b) there are easy examples of conservative and dissipative systems satisfying the hypotheses *I,II* above, and [4] b1) in stationary (non)equilibrium,

b2) with equal or disjoint attracting and repelling surfaces,

b3) time reversible,

b4) with as many degrees of freedom and negative Lyapunov exponents as wished (unrelated to the number of positive ones)

b5) and whose stationary distributions are explicitly and completely constructed,

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The spherical chickens remind us of the words against Galileo, more elegant but equally rough, if obscure: **Viri Galilei, quid statis adspicientes in coelum?**, (T.Caccini, 1614, a proposito della matematica, arte diabolica e fautrice di eresia:)

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