Viscosity, Reversibility, Statistical Ensembles: in a Navier-Stokes fluid

What is viscosity? there is no such force at microscopic level! Since Maxwell (1866) it arises as an average of microscopic events. The "dramatic" effect $\rightarrow$ irreversibly.

Could basic fluid equations be modified, becoming reversible and still agree with experiment?

To modify basic equations "saving the results" is quite common. Typical is equivalence between microcanonical ensemble (with constant energy or Hamiltonian eqs.) equations) and isokinetic ensemble (with constant kinetic energy or Gaussian eqs.)

As a simple paradigmatic case (of much more general systems) look at a fluid in a periodic container, incompressible, subject to a constant force.

Equilibrium states $\Rightarrow \mathcal{N} = \rho V$ particles in volume V and interaction potential $U(\mathbf{q}) \Rightarrow$ probability distributions determining average values of many observables

1) *i.e.* local observables $O \in \mathcal{O}_{loc}$: $O(\mathbf{p}, \mathbf{q})$, depend on $q_i \in \mathbf{q}$ located in regions $\Lambda \subset V$.

2) distributions depend on equations of motion.

Which among the invariant prob. distr. is the correct one? For isolated systems Ergodic Hypothesis (EH) provides (a) solution: for a.a. data $\mathbf{u} = (\mathbf{p}, \mathbf{q})$

$$\mu_E(d\mathbf{p}d\mathbf{q}) = \frac{1}{Z}\delta(H_V(\mathbf{p}, \mathbf{q}))d\mathbf{p}d\mathbf{q}$$

As the energy $E = eV$ varies the distributions are collected in $\mathcal{E}_E^{mc,V}$, microcanonical ensemble.

Why? data are always generated randomly with a unknown distribution which however is (taciltly ?) assumed of the form $\rho(\mathbf{u})d\mathbf{u}$.

If system is chaotic (e.g. hyperbolic) it is a theorem that a.a data $\mathbf{u}$ evolve visiting sets with well defined frequency *independent* of the unknown distribution for the data generation, called the SRB distribution.

This asymptotic behavior only depends on the hyperbolicity of the motion on phase space or, in the case of dissipative evolution, on the attracting set.

The two remarks contain the essence of Ruelle's proposal:

"the initial data are random with distr, $\rho(\mathbf{u})d\mathbf{u}$ (unknown but absolutely cont.) and motions are "generically" chaotic so that the statistics of the motions is uniquely determined as the SRB distribution"

If account is taken that only few observables are physically interesting then other distributions might provide the same averages for the interesting observables, particularly in the case of macroscopic systems, and can be collected in other "ensembles" $\mathcal{E}_\beta^V$. For instance the canonical distrib.

$$\mu_\beta^{c,V}(d\mathbf{p}d\mathbf{q}) = \frac{1}{Z} e^{-\beta H_V(\mathbf{p},\mathbf{q})} d\mathbf{p}d\mathbf{q}$$

The apparent resulting ambiguity is solved, in equilibrium, by the equivalence between distrib. in $\mathcal{E}_E^{mc,V}$ and $\mathcal{E}_\beta^{c,V}$: "canonical distribution $\mu_\beta^V \in \mathcal{E}^c$ is equivalent to the microcanoical $\widetilde{\mu}_E^V \in \mathcal{E}^{mc}$ if β, E are s.t.

$$\mu_\beta^V(H_V(\mathbf{p},\mathbf{q})) = E \;\Rightarrow\; \lim_{V\to\infty} \mu_\beta^V(O) = \lim_{V\to\infty} \widetilde{\mu}_E^V(O)$$

and μ's are "*equivalent* in the thermodynamic limit".

Ruelle's generalization of EH unifies equilibrium and nonequilibrium and assumes that generically chaotic systems are such in the precise sense of Axiom A implies also in nonequilibrium $\Rightarrow$ unique statistics for the stationary states.

Axiom A can be simplified (Cohen & G) replacing it with "generically" chaotic motions evolve towards a smooth attracting surface over which motion is chaotic in the sense of Anosov (stronger than Axiom A): Chaotic hypothesis or CH.

Hence is natural to ask whether a theory of ensembles (*i.e.* of families of distr.) in 1-to-1 correspondence yield equivalent statistical descriptions for same stationary state.

Any theory of large (macrosc.) systems $\Rightarrow$ requires

(1) Regularization of equations (via a "cut off")

(2) Restriction on observables ("local observables")

Regularization, necessary in essentially all cases, replaces $\dot{\mathbf{u}} = f_R(\mathbf{u})$ (∞-dim) by a regularized $\dot{\mathbf{u}} = f_R^V(\mathbf{u})$ ($< \infty$-dim).

Stationary $\mu_R^V(d\mathbf{u})$ uniquely determined by Ruelle's extension of ergodic hypothesis (*i.e.* SRB distrib.).

Form a family $\mathcal{E}_R^V$ of distributions assigning average values to the restricted observables.

For instance:

(a) In Stat. Mech: local observables and cut-off $V =$ container size; $\Rightarrow$ find their averages at limit as $V \to \infty$:

(b) In Fluid Mech.: large scale observables (*i.e.* functions of velocities with "waves" $|\mathbf{k}| < K \ll N$) and cut-off N on the maximum wave $|\mathbf{k}|$: $\Rightarrow$ find averages at limit as $N \to \infty$

Concentrate on the paradigmatic case of periodic NS fluid, [1, 2].

(a) 2/3-Dim., incompressible,

(b) **fixed large scale forcing** F (*e.g.* **with only one or few** Fourier's waves and $\|F\|_2 = 1$),

(c) dissipate heat via viscosity $\boldsymbol{\nu} = \frac{\mathbf{1}}{\mathbf{R}}$

NS_{irr}: $\dot{u}_\alpha = -(\mathbf{u} \cdot \boldsymbol{\partial})u_\alpha - \partial_\alpha p + \frac{1}{R}\Delta \mathbf{u}_\alpha + F_\alpha, \qquad \partial_\alpha u_\alpha = 0$

Velocity: $\mathbf{u}(x) = \sum_{\mathbf{k}\neq\mathbf{0}} u_\mathbf{k} \frac{i\mathbf{k}^\perp}{|\mathbf{k}|} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \overline{u}_\mathbf{k} = u_{-\mathbf{k}}$ (NS-2D)

$NS_{2,irr}$: $\dot{u}_\mathbf{k} = -\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} \frac{(\mathbf{k}_1^\perp \cdot \mathbf{k}_2)(\mathbf{k}_2^2 - \mathbf{k}_1^2)}{2|\mathbf{k}_1||\mathbf{k}_2||\mathbf{k}|} u_{\mathbf{k}_1} u_{\mathbf{k}_2} - \nu \mathbf{k}^2 u_\mathbf{k} + f_\mathbf{k}$

"Regularize eq.": waves $|\mathbf{k}_j| \le N$. At UV-Cut-off , N.

Remark: $I u_\alpha = -u_\alpha$ implies solutions $t \to S_t^{irr}\mathbf{u}$ s.t.

$$I S_t^{irr} \neq S_{-t}^{irr} I \qquad \Rightarrow \ \textbf{irreversibility}$$

Given init. data u, evolution $u \to S_t^{irr} u$ generates a steady state (*i.e.* a SRB probability distr.) $\mu_R^{irr,N}$ on M_N.

Unique out a 0-volume of u's, for simplicity AT R small: "NS gauge symmetry" exists.; phase transitions, [3, 4, 5].

As R varies steady distr. $\mu_R^{irr,N}(du)$ are collected in $\mathcal{E}^{irr,N}$:

This is A statistical ensemble of stationary nonequilibrium distrib. for NS_{irr}.

Average energy E_R, average dissipation En_R, Lyapunov spectra (local and global) ... will be defined, *e.g.*:

$$E_R = \int_{M_N} \mu_R^{irr,N}(du)||u||_2^2, \qquad En_R = \int_{M_N} \mu_R^{irr,N}(du)||\mathbf{k}u||_2^2$$

Are there other ensembles whose elements can be put in $1-1$ correspondence with the ones in $\mathcal{E}^{irr,N}$ and give same average values to "local obsevables"?

Consider new equation, NS_{rev} (with cut-off N):

$$\dot{\mathbf{u}}_{\mathbf{k}} = -\sum_{\mathbf{k_1+k_2=k}} \frac{(\mathbf{k}_1^{\perp}\cdot\mathbf{k}_2)(\mathbf{k}_2^2-\mathbf{k}_1^2)}{2|\mathbf{k}_1||\mathbf{k}_2||\mathbf{k}|}\mathbf{u}_{\mathbf{k}_1}\mathbf{u}_{\mathbf{k}_2} - \alpha(\mathbf{u})\mathbf{k}^2\mathbf{u}_{\mathbf{k}} + f_{\mathbf{k}}$$

with α **s. t.** $\mathcal{D}(u) = ||\mathbf{k}u||_2^2 = En$ (the enstrophy)is **exact const of motion** on $u \to S_t^{rev}u$.:

$$\Rightarrow\ \alpha(u) = \frac{\sum_{\mathbf{k}}\mathbf{k}^2 F_{-\mathbf{k}}u_{\mathbf{k}}}{\sum_{\mathbf{k}}\mathbf{k}^4|u_{\mathbf{k}}|^2} \qquad e.g.\ \ D=2$$

New eq. is reversible: $IS_t^{rev}u = S_{-t}^{rev}Iu$ (as α is odd).

α is "a reversible viscosity"; (if $D=3$ α is ∼different)

Rev. eq. is an empirical model of "thermostat" on the fluid and should (**?**) have **same effect** of empirical constant friction (that can also be a thermostat model).

NS_{rev} generates a family of steady states $\mathcal{E}^{rev,N}$ on M_N: $\mu^{rev,N}_{En}$ parameterized by constant value of **enstrophy** En.

$\alpha(u)$ in NS_{rev} **will wildly fluctuate** at large R (*i.e.* small viscosity ν) thus "**self averaging**" to a const. value ν "homogenizing" the eq. into NS_{irr} with viscosity ν.

Equivalence mechanism by analogy with Stat. Mech.

(1) analog of "local observables": functions $O(u)$ which depend only on $u_{\mathbf{k}}$ with $|\mathbf{k}| < K$. "Locality in momentum"

(2) analog of "Volume": just the cut-off N confining the $\mathbf{k}$

(3) analog of "state parameter": viscosity $\nu = \frac{1}{R}$ (irrev. case) or enstrophy En (rev. case).

$$\text{Equivalence condition}: \ \mu^{rev,N}_{En}(\alpha) = \frac{1}{R}$$

Equivalence is **conjectured** at $N = \infty$ in analogy with the thermodynamic limit $V \to \infty$, for **all** R.

Averages of large scale observables: same as $N \to \infty$ for

$$\mu^{irr,N}_{R} \in \mathcal{E}^{irr,N} \ \textbf{and} \ \ \mu^{rev,N}_{En} \in \mathcal{E}^{rev,N}$$

provided, $\mathcal{D}(\mathbf{u}) \stackrel{def}{=} \sum_{\mathbf{k}} \mathbf{k}^2 |\mathbf{u}_{\mathbf{k}}|^2$ is s.t.

$$\mu^{irr,N}_{R}(\mathcal{D}) = En, \qquad \text{or} \qquad \mu^{rev,N}_{En}(\alpha) = \frac{1}{R} = \nu$$

Balance: multiplying NS eq. by $\overline{u}_{\mathbf{k}}$ and sum on $\mathbf{k}$:

$$\frac{1}{2}\frac{d}{dt}\sum_{\mathbf{k}}|u_{\mathbf{k}}|^2 = -\boldsymbol{\gamma}\mathcal{D}(\mathbf{u}) + W(\mathbf{u}), \quad \boldsymbol{\gamma} = \nu \textit{ or } \alpha(\mathbf{u})$$

(transport terms $= 0$, $D = 2, 3$), $\mathcal{D}(\mathbf{u}) = \sum_{\mathbf{k}} \mathbf{k}^2 |\mathbf{u}_{\mathbf{k}}|^2 =$ enstrophy and $W = \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}} \mathbf{u}_{-\mathbf{k}} =$ power of external force.

Remark that W is a local observable (since $\mathbf{f}$ is such)

Hence time averaging

$$\frac{1}{R}\mu_R^{irr,N}(\mathcal{D}) = \mu_R^{irr,N}(W), \qquad \mu_{En}^{rev,N}(\alpha) En = \mu_{En}^{rev,N}(W)$$

But W is local (as $\mathbf{f}$ is such) and, if the conjecture holds, has equal average under the equivalence condition: hence $\mu_R^{irr,N}(\mathcal{D}) = En$ **implies** the relation

$$\lim_{N\to\infty} R\mu_{En}^{rev,N}(\alpha) = 1$$

This becomes a first rather stringent test of the conjecture.

But it will be useful to pause to illustrate a few prelimnary simulations and checks.

Unfortunately the following simulations are in dimension 2 ($D = 3$ is at the moment beyond the available (to me) computational tools) although present day available NS codes should be perfectly capable to perform detailed checks in rapid time, [6].

Concentrate on the first test:

$$\lim_{N\to\infty} R\mu_{En}^{rev}(\alpha) = 1$$

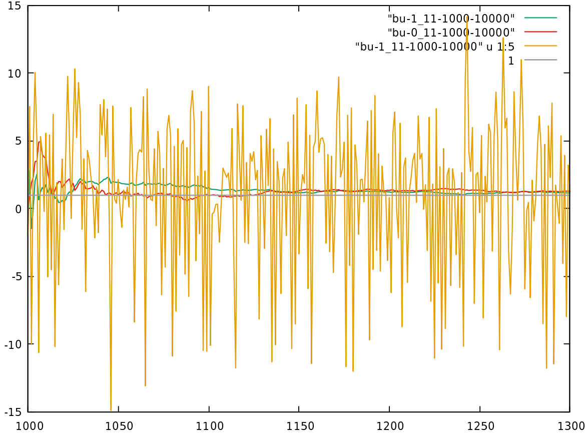

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Fig.1 (detail): Running average of reversible friction $R\alpha(u) \equiv R\frac{2Re(f_{-\mathbf{k}_0}u_{\mathbf{k}_0})\mathbf{k}_0^2}{\sum_{\mathbf{k}}\mathbf{k}^4|u_{\mathbf{k}}|^2}$, superposed to conjectured 1 and to the fluctuating values of $R\alpha(u)$. **Initial transient** $t < 1000$. Evol.: NS_{rev}, **R=2048**, $\mathcal{N} = 960$ modes, Lyap. $\simeq 2$, x-unit $= 2^{19}$ step of size 2^{-17}.

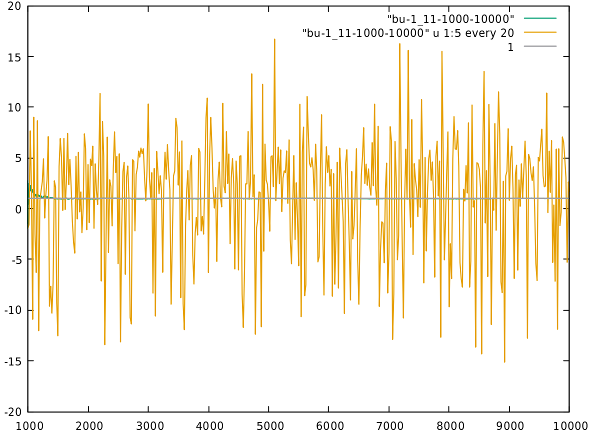

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Fig.1-bis: As previous fig. but time 10 times longer: data reported “every 20”, with visual aid at 1.

The following Fig.2 (similar to Fig.1 but w. NS_{irr}):

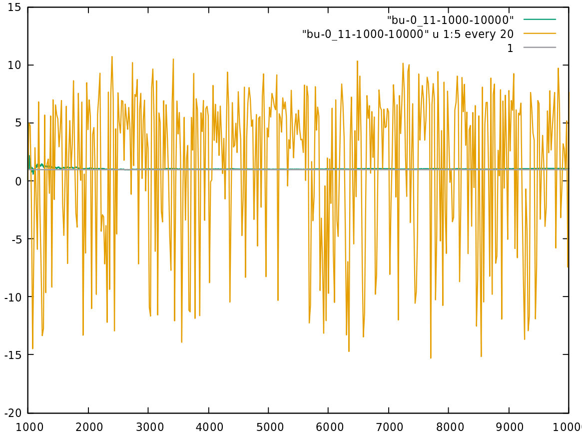

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Fig.2: As Fig.1 but running average of reversible friction $R\alpha(\mathbf{u})$ regarded as observ. in NS_{irr}, superposed ro value 1 and to fluctuating values of $R\alpha(\mathbf{u})$. An extension ? of conjecture since $\alpha(\mathbf{u})$ is not local.

The similarity of fluctuations ⇒ possibility that via fluctuations new properties about NS_{irr} might be obtained (and answer the common objection **so what?**).
Idea: Fluctuation theorem (FT) could apply even to NS_{irr}.
Check the "Fluctuation Relation" in the **reversible** evolutions: for the divergence (trace of the Jacobian) $\boldsymbol{\sigma}(u) = -\sum_{\mathbf{k}} \partial_{u_{\mathbf{k}}} (\dot{u}_{\mathbf{k}})_{rev}$: let p (time τ average of $\frac{\sigma}{\langle\sigma\rangle}$)

$$p \stackrel{def}{=} \frac{1}{\tau}\int_0^\tau \frac{\boldsymbol{\sigma}(\mathbf{u}(t))}{\langle\,\boldsymbol{\sigma}\,\rangle_{rev}} dt,$$

then a theorem for Anosov systems:

$$\frac{P_{srb}(p)}{P_{srb}(-p)} = e^{\tau\,\mathbf{1}\,\mathbf{p}\,\langle\,\boldsymbol{\sigma}\,\rangle_{\mathbf{rev}}} (\text{sense of large deviat. as } \tau \to \infty)$$

$$or\;\; P_{srb}(p) \stackrel{def}{=} c e^{\tau s(p) + o(\tau)}, \qquad \frac{\mathbf{s}(\mathbf{p}) - \mathbf{s}(-\mathbf{p})}{\boldsymbol{\tau}\langle\,\boldsymbol{\sigma}\,\rangle_{\mathbf{rev}}} = \mathbf{p} + \mathbf{o}(\tau)$$

Replacing reversible evolution with irreversible but studying the reversible divergence becomes a "*reversibility test on the irreversible flow*"!

Can this be applied to turbulence **?**

First: is the attracting surface the full phase space **?**

This may be true *only* at strong regularization.

If so the system NS_{rev} would be Anosov (by CH) and reversible: hence number of Lyap. exp which are ≥ 0 should equal that of the < 0

So begin with regularization: $\mathcal{N} = 48$ modes.

The calculation of the **local** Lyap. exp. is easier: at $\mathbf{u}$ are the Lyap. exp. of the linear flow $\mathbf{v} \to e^{tJ(\mathbf{u})}\mathbf{v}$ with $J(\mathbf{u})$ the linearization matrix at $\mathbf{u}$.

The surprising result is:

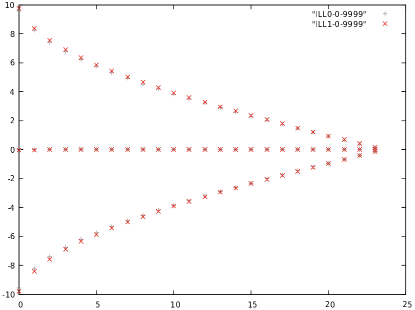

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Fig.3: $R = 2048, \mathcal{N} = 48$ modes. The local Lyapunov exponents for **both** NS_{rev} **and** NS_{irr}. This shows coincidence on the graph scale and equal number of exponents of either sign. Hence by CH the NS_{rev} at least should show a FR verifying the Fluctuation theorem. The surprise is the pairing $\lambda_k + \lambda_{\mathcal{N}-1-k} \simeq$ constant middle values, **warning: but this is not exact**.

What about FR ? one can imagine that FR holds for NS_{rev} for the divergence $p = \frac{1}{\tau}\int_0^\tau \frac{\sigma(S^t_{rev}\mathbf{u})}{\sigma_+}$. It does

And about NS_{irr}? of course the divergence is $\nu\sum_{\mathbf{k}}\mathbf{k}^2$ constant. However the divergence of the NS_{rev} is $\sigma(\mathbf{u}) = \sum_{\mathbf{k}}\frac{\partial\dot{u}_{\mathbf{k}}}{\partial u_{\mathbf{k}}}$, easily computed as:

$$\sigma(\mathbf{u}) = \alpha(\mathbf{u})\Big(2K_2 - 2\frac{E_6(\mathbf{u})}{E_4(\mathbf{u})}\Big) + \frac{F(\mathbf{u})}{E_4(\mathbf{u})}$$

$$2K_2 = \sum_{\mathbf{k}}\mathbf{k}^2,\ E_4(\mathbf{u}) = \sum_{\mathbf{k}}(\mathbf{k}^2)^2|\mathbf{u}_{\mathbf{k}}|^2,$$

$$E_6(\mathbf{u}) = \sum_{\mathbf{k}}(\mathbf{k}^2)^3|\mathbf{u}_{\mathbf{k}}|^2,\ F(\mathbf{u}) = \frac{\sum_{\mathbf{k}<N}(\mathbf{k}^2)^2\overline{f}_{\mathbf{k}}\mathbf{u}_{\mathbf{k}}}{E_4(\mathbf{u})}$$

Imagine $\sigma(\mathbf{u})$ as an observable for NS_{irr}: **idea** is that $p = \frac{1}{\tau}\int_0^\tau \frac{\sigma(S^t_{irr}\mathbf{u})}{\sigma_+}dt$ **verifies FR.**

The check works for both NS_{rev} **and** NS_{irr}.

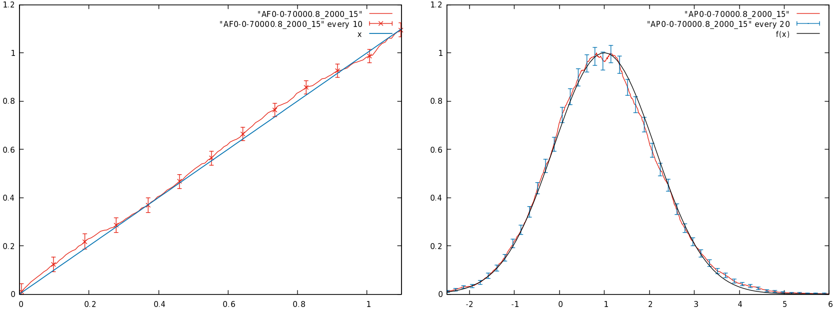

Fig.4,5: Left: red curve is the graph of $(s(p) - s(-p))/\tau\sigma_+$ with error bars (every 10), the blue straight line is a visual aid. The data are $7 \cdot 10^4$ averages over τ iterations over initial data collected every $4\tau/h$ steps with $\tau = 8, h = 2^{13}$. On the right: histogram (red curve) with error bars of the p distribution normalized to have maximum 1. Black line: Gaussian fit $f(x) = e^{-a(p-1)^2}$ to red line. **FT ok**.

However if N is larger:

Problem 1: if attracting set $\mathcal{A}$ has lower dimension, time reversal symmetry I cannot be applied because $I\mathcal{A} \neq \mathcal{A}$. This certainly occurs if N becomes large enough, [7, 8].

Help could come **if** exists further symmetry P between $\mathcal{A}$ and $I\mathcal{A}$ *commuting* with S_t: $PS_t = S_tP$.

Then $P \circ I : \mathcal{A} \to \mathcal{A}$ becomes a time reversal symmetry of the motion restricted to $\mathcal{A}$.

Problem 2: even supposing existence of P, still **is is not** possible to apply FR because, at best, it would concern the contraction $\sigma_{\mathcal{A}}(\mathbf{u})$ of $\mathcal{A}$ and not the $\sigma(\mathbf{u})$ of M_V.

The $\sigma(\mathbf{u})$ receives contributions from the exponential approach to $\mathcal{A}$: which obviously do not contribute to $\sigma_{\mathcal{A}}$.

How to recognize such contributions ?

Help for both problems could come from "pairing property"
Often Lyapunov exps (local and global) arise in pairs with almost constant average or average on a regular curve.

In a few systems pairs have an exactly constant average.

For NS an idea can be obtained from the local exponents (eigenvalues of the symmetric part of Jacobian matrix).

For instance NS seems to enjoy a pairing rule: the following figure illustrates it (at strong regularization and large Reynolds number); looks close to exact (at graph scale!).

Pairing in NS cannot stay at all regularizations: the above should be considered due to the small N. However it has been proposed that the apparent "pairing line" becomes at large N a "pairing curve" (as it appears so at small R).

Examine the local Lyap. exponents at larger $\mathcal{N}$:

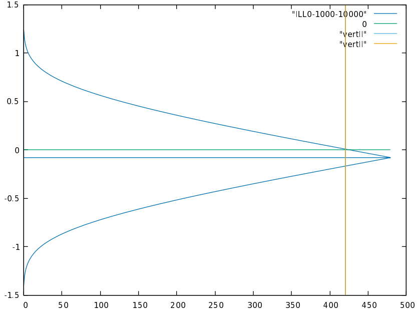

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Fig.6: $R = 2048, \mathcal{N} = 960$, **local** exponents ordered by decreasing: λ_k, $0 \le k < \mathcal{N}/2$, by increasing: $\lambda_{\mathcal{N}-1-k}, 0 \le k < \mathcal{N}/2$, the line $(\lambda_k + \lambda_{\mathcal{N}-1-k})$ and the line $\equiv 0$. Irreversible case and apparent pairing rule and dimensional loss $\varphi \simeq \frac{430}{480}$.

In Anosov reversible systems the number of positive exp. = number of negative: hence proposal, [9, p.445],[10],

"attracting surface $\mathcal{A}$ dimension = twice the number of positive exponents" and

if pairing: twice num. of opposite sign pairs.

Implication: $\sigma_{\mathcal{A}}(\mathbf{u})$ is proportional to the total $\sigma(\mathbf{u})$ if pairing to a constant

$$\sigma_{\mathcal{A}}(\mathbf{u}) = \varphi\sigma(\mathbf{u}), \qquad \varphi = \frac{number\ of\ opposite\ pairs}{total\ number\ of\ pairs}$$

(if pairing to a curve $\sigma_{\mathcal{A}}(u) = \sigma(u) + \sum_{pairs<0}(\lambda_j + \lambda'_j)$). **Why?**

Idea: negative pairs correspond to exponents associated with attraction to $\mathcal{A}$: hence irrelevant in computing $\sigma_{\mathcal{A}}$.

Still FT requires time reversal symmetry on the attracting set. So eistence of P is required.

Assuming **existence of** P, hence time-reversal on attractor, and **pairing** + CH $\Rightarrow$ FR **but slope** $\boldsymbol{\varphi} < 1$:

$$\frac{s(p) - s(-p)}{\tau\langle\,\sigma\,\rangle} = p\boldsymbol{\varphi}, \quad \text{rather than} \quad p: \quad \text{in fig. } \boldsymbol{\varphi} \simeq \frac{\mathbf{420}}{\mathbf{460}}$$

If true: this will be a "check of reversibility" in NS_{irr}.

More elaborate checks are being attempted: [6, 11] +

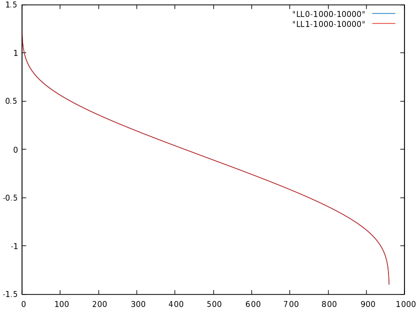
(a) moments of large scale observables rev & irr

(b) local Lyap. exps of matrices different from Jacobian

(c) check of the fluctuation rel., particularly in irrev. cases, (shown above to be accessible already with 960 modes and $R = 2048$): $\Rightarrow$ FR with slope $\boldsymbol{\varphi} < 1$ (Axiom C ?), [9, 12].

(d) More values of R and N an example with R larger than in the preceding cases yields similar results (not shown).

Details on the above Fig.6 Lyapunov exponents:



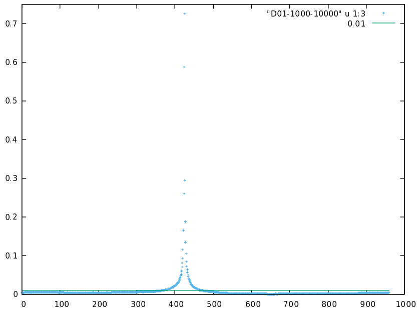


Fig.7,8: LLyap. exp. NS_{rev} & NS_{irr}, $R = 2048, \mathcal{N} = 960$, computed every $4/h$ steps with $h = 2^{-13}$ and averaged over 6000 data. The spectra overlap on he scale of the graph. The right panel shows the difference $\frac{|\lambda_k^{irr} - \lambda_k^{rev}|}{\max(\lambda_k^{irr}|,|\lambda_k^{rev}|)}$, $k = 0, \ldots, \mathcal{N}$; the bar marks the 1% difference.

Unexpected: fluctuations in NS_{irr} & NS_{rev} of Fig.7

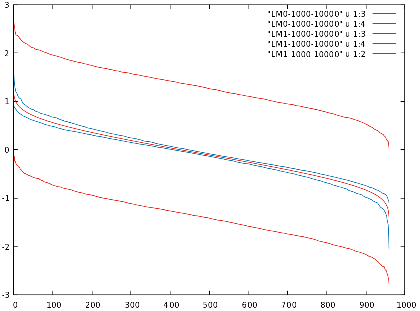


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Fig.9: Fluct. max.-min. showing the $\mathcal{N} = \mathbf{960}$ exponents. Central lines rev & irr. superposed (averaged over 800 samples) with coincidence of NS_{rev} with the NS_{irr} exponents. Strong fluct. between max-min variations (upper and lower lines) in NS_{rev}. Average reversible (red) enclosed between the two (blue) lines containing the average exp. of NS_{irr}.

Example of moments of local observables:

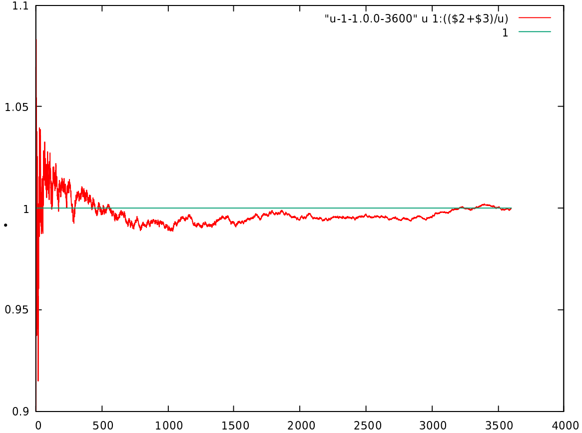

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Fig.10: Running averages in NS_{rev} of $(|Re\, u_{11}|^4 + |Im\, u_{11}|^4)/\langle\, |Re\, u_{11}|^4 + |Im\, u_{11}|^4\, \rangle_{irr}$, $R = 2048$, 960 modes. Conjecture **yields** ratio tending to 1

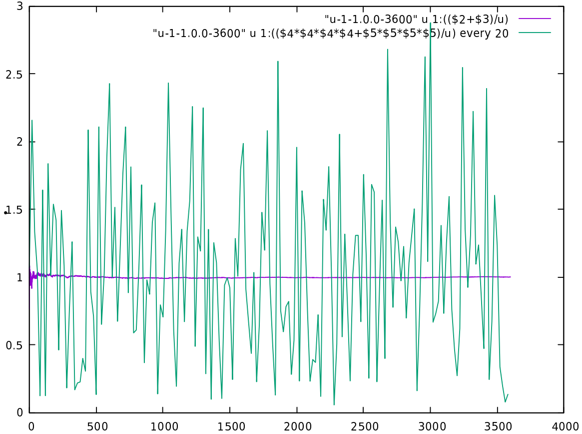

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Fig.11: Same running averages in NS_{rev} of $(|Re\,u_{11}|^4 + |Im\,u_{11}|^4)/\langle\,|Re\,u_{11}|^4 + |Im\,u_{11}|^4\,\rangle_{irr}$, and their rev. fluctuations, for $R = 2048$, 960 modes.

Concluding the simulation

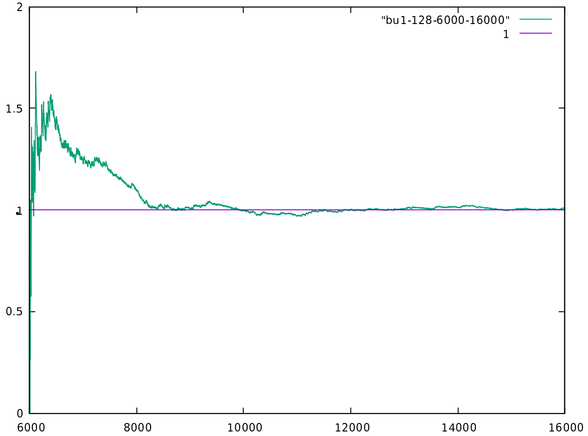

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Fig.12: Illustration of the conjecture on a $\mathcal{N} = 3968$ modes NS: the running averages of $R\alpha$ in the reversible NS should tend to 1, according to conjecture.

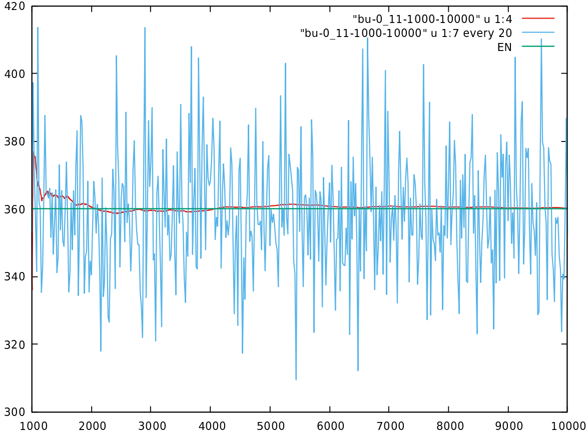

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Fig.13: NS_{irr}: **Running** average of the work $R\sum_{\mathbf{k}} F_{-\mathbf{k}} u_{\mathbf{k}}|$ (**violet**) in NS_{rev}; and **convergence** to average enstrophy En (**orange** straight line),
blue is running average of enstrophy $\sum_{\mathbf{k}} \mathbf{k}^2 |u_{\mathbf{k}}|^2$ in NS_{irr}, enstrophy **fluctuations** violet in NS_{irr}: $R = 2048, \mathcal{N} = 960$.

What happens in cases in which the attractor is not chaotic. As said the proposal remains the same.

A simple check is the case in which $f_{\mathbf{k}} \neq 0$ only for $\mathbf{k} = \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{k} = \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

In this case Marchioro has proved that the system admits a global attracting fixed point $\forall R$, *i.e.* a simple laminar motion.

And the proposal is very precisely confirmed: all Lyapunov exp. are < 0 and the evolution leads to Marchioro's solution, [5] with $R\langle\, \alpha \,\rangle_0^t \xrightarrow[t\to\infty]{} 1$.

There are other cases in which the system has several attractors which should be studied: this is more difficult because the evolution might be unstable and the evolution could spend time near one attractor or another and the precision will play a key role.

Finally rigorous estimate of number $\mathbf{n_1}$ of Lyap. exp. needed so that their sum remains > 0 ($\sim$ KY-dimension):

$$\mathbf{n_1} \le \sqrt{2} A (2\pi)^2 \sqrt{R} \sqrt{R\, En}, \qquad A = 0.55..$$

in 2D, while in 3D a similar estimate holds but it involves a norm different from enstrophy. (Ruelle if $d = 3$ and Lieb if $d = 2, 3$, [13, 8]).

Applied here it would require $\mathcal{N} \sim 2.10^4$ for NS2D: not accessible in the simulations presented here but not impossible in principle with computers and computation methods already available, at least if $D = 2$.

There seems to be no estimate of the number $\mathbf{n_0}$ of > 0 eigenvalues in 2D, 3D in terms of the enstrophy. **Question:** is it possible that $\mathbf{n_0}$ could be bounded by $R\sqrt{En}$ times a N-dependent constant (a power of $\log N$ in 2D?).

Finally further careful checks are required, particularly since inspiring ideas are, to say the least, **controversial** as shown by quotes from a well known treatise, [14, p.344-347] and [2, app.A].

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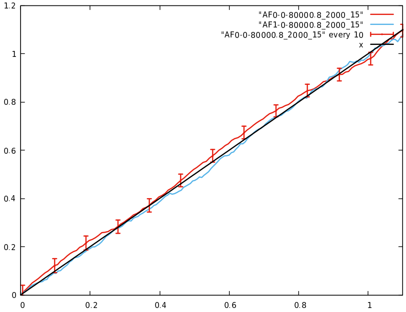
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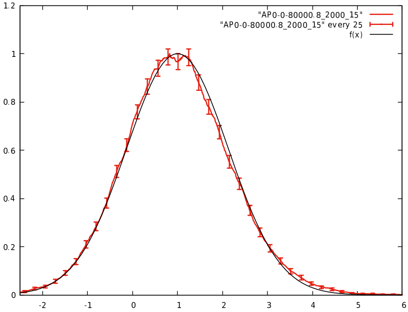
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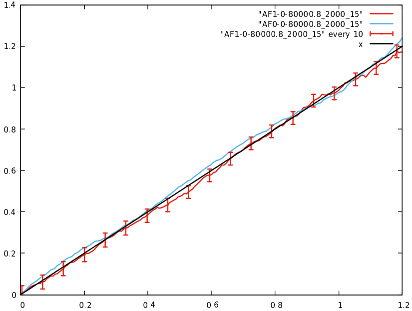
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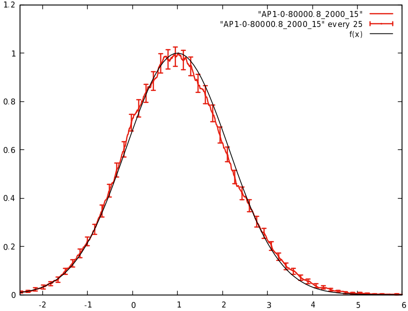
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