Viscosity, reversibility, chaotic hypothesis in NS fluids

In SM several invariant stationary states depending on parameters like total energy E, total kinetic energy T, or parameters that fix average values of other observables. describe stationary states.

Such stationary distributions are (often) shown equivalent: they lead to the same "Thermodynamics" in the limit of infinite volume, if limited to study properties of a restricted class of observables: the so called local observables.

In the theory of fluids a situation arises which bears some resemblance with the above. It will be examined mainly in the case of an incompressible fluid in a periodic container.

The velocity field $\mathbf{u}(\mathbf{x})$, $\mathbf{x} \in \Omega = [0, 2\pi]^d$, $d = 2, 3$ is:

$$\mathbf{u}(\mathbf{x}) = \sum_{0<|\mathbf{k}|\le N} \mathbf{u}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{k} = (k_\beta)_{\beta=1,\dots d} \in Z^d$$

$$\mathbf{u}_{\mathbf{k}} = \sum_{\beta=1,2} i\, u_{\beta,\mathbf{k}}\, \mathbf{e}_\beta(\mathbf{k}), \quad \mathbf{k} \cdot \mathbf{e}_\beta(\mathbf{k}) = 0$$

and the incompressible NS equations:

$$\dot{\mathbf{u}}(x) = -(\underset{\sim}{\mathbf{u}}(x) \cdot \boldsymbol{\partial}_{\mathcal{X}})\mathbf{u}(x) - \nu \boldsymbol{\Delta} \mathbf{u}(x) - \boldsymbol{\partial}_x P(x) + \mathbf{f},$$

$$\boldsymbol{\partial}_x \cdot \mathbf{u}(x) = 0, \quad \int_{\mathcal{T}^d} dx\, \mathbf{u}(x) = \vec{0}$$

become equations, *“INS=irreversible NS”*, for the velocity harmonics $u_{\beta,\mathbf{k}}$.

Here too no general existence-uniqueness (in dimension 3).

The main difficulty is the lack of control of the very small scales features of the velocity fields.

Here interest is directed towards properties of large classes of observables; but far from all, and which do not depend on arbitrarily small scale structures.

The existence problem is analogous to the problem of existence of solutions of the equations of motion for infinite particle systems: and, as in that case, here will be simply set aside by imagining that the equations only regard the velocity components $u_{\beta,\mathbf{k}}$ with $|\mathbf{k}| < N$: the UV cut-off N becomes analogous to the volume cut-off V in SM.

Idea: as in the SM case infinite systems are an abstraction, so in FM is $N = \infty$

Of course one is **NOT interested** in the value of N, much as in SM in the size V of the volume: it should not be relevant (as long as it is large compared to the scale of the observables). This leads to propose

1) define "local" observables the $O(\mathbf{u})$ which depend on $\mathbf{u}$ only through Fourier's components $u_{\beta,\mathbf{k}}$ with $|\mathbf{k}| < K$: they are supposed to evolve in time with the NS equations with UV cut-off $N \gg K$: $t \to \mathbf{u}^N(t)$

2) define a "stationary state" of the fluid the collection of the limits $N \to \infty$ of the time averaged values $\langle\, O \,\rangle_N$, along the evolution $t \to \mathbf{u}^N(t)$), of the local O's:

$$\langle\, O \,\rangle = \lim_{N\to\infty} \langle\, O \,\rangle_N$$

Here $K \ll N$ is arbitrary but fixed and observables O are alike the ones in SM depending only on particles located in a finite volume $K \ll V$.

Summarizing the parallel SM$\sim$FL:

SM : (a) local observables O depend only on particles in a finite region $K \subset V$ (arbitrary, V-independent).
(b) time evolution $\vec{x} \to \vec{x}(t)^V$ depends on vol. cut-off V.

FM : (a) local obs. O depend only on modes $\mathbf{u}_{\beta,\mathbf{k}}$ with $\mathbf{k}$ in a finite $K < N$ (arbitrary, N-independent).
(b) time evolution $\mathbf{u} \to \mathbf{u}(t)^N$ depends on UV cut-off N.

In both cases interest is on "stationary states", *i.e.* on the collection of the time averages $\langle\, O\,\rangle^V$ or, resp., $\langle\, O\,\rangle^N$ in the limits $V, N \to \infty$, (in the simple cases of no phase transitions, resp, no intermittency, *i.e.* single attractor).

Given the analogy between equilibrium states in SM and the above vision of the stationary states of NS fluids, it i's natural to ask if there is room to define other families of averages of local observables, physically equivalent to the above considered.

As pointed out in [1] turbulence should be considered a problem of non-equilibrium SM.

Other ensembles possibility and their equivalence arises if NS equations are regarded as a Hamiltonian system (Euler equations) subject to forces and to constraints (“thermostats”) that impose incompressibility (*i.e.* prescribe pressure versus temperature at fixed density).

Viscosity is then an empirical quantity that accounts for the thermostat action.

Then it should be possible to replace ν with another empirical quantity achieving the same aim of allowing the evolution to generate a stationary state:

which on the selected observables will attribute the same averages in the continuum limit $N \to \infty$.

A similar situation arises in equilibrium SM: the same state, in the thermodynamic limit, is achieved for the local observables in the microcanonical cases (evolution at fixed total Energy) or in the isokinetic cases (evolution at fixed total Kinetic energy) ...

A large number of different equations, associated with different thermostats, can be imagined to generate the same collections of average values, at fixed external parameters (*i.e.* forcing).

Concentrating on stationary NS fluids, the physical role of the thermostat is to avoid "blow up", due to power injected

$$\langle\, W\,\rangle = \langle\, \int_\Omega \mathbf{f}(\xi)\cdot\mathbf{u}(x)^N dx\,\rangle$$

allowing to reach a stationary state.

So we naturally classify stationary states by the time average of the "dissipation" $\langle\, W\,\rangle$ which in the standard NS equation is proportional to the average "enstrophy"

$$En = \langle\, \int_\Omega (\partial\mathbf{u}(x))^2 dx\,\rangle : \quad \nu En = \langle\, W\,\rangle$$

A proposal for an alternative thermostat is to replace $\nu\Delta\mathbf{u}$ by $\alpha(\mathbf{u})\Delta\mathbf{u}$ with $\alpha(\mathbf{u})$ a multiplier such that the enstrophy En itself is a constant of motion. The equation is

$$\dot{\mathbf{u}} = -(\underset{\sim}{\mathbf{u}}\cdot\underset{\sim}{\boldsymbol{\partial}})\mathbf{u} + \alpha(\mathbf{u})\Delta\mathbf{u} + \mathbf{f} - \boldsymbol{\partial}P$$

with, if $\Lambda(\mathbf{u}) = -\int_\Omega(\underset{\sim}{\mathbf{u}}\cdot\underset{\sim}{\boldsymbol{\partial}}\mathbf{u})\cdot\Delta\mathbf{u}\,dx$,

$$\alpha(\mathbf{u}) = \frac{\Lambda(\mathbf{u}) + \sum_{\mathbf{k}}\mathbf{k}^2\mathbf{f}_{\mathbf{k}}\cdot\overline{\mathbf{u}}_{\mathbf{k}}}{\sum_{\mathbf{k}}\mathbf{k}^4|\mathbf{u}_{\mathbf{k}}|^2},\ d=3$$

will be named "RNS"=reversible NS (as $\alpha(-\mathbf{u}) = -\alpha(\mathbf{u})$) with UV cut-off at $|\mathbf{k}| \le N$.

Given a $\mathbf{f}$ "large scale forcing" , $\mathbf{f}_{\mathbf{k}} \neq 0, |\mathbf{k}| < k_0, \|\mathbf{f}\| = 1$ the equation in $\Omega = [0, 2\pi]^d$ has only one parameter: the viscosity ν for INS and the enstrophy En for RNS.

Fixed the UV cut-off N, as ν or En vary let

$\mu_\nu^{i,N}(d\mathbf{u})$= stationary distribution for INS
$\mu_{En}^{r,N}(d\mathbf{u})$= stationary distribution for RNS

and

"viscosity ensemble" be collection $\mathcal{E}^{i,N}$ of the $\mu_\nu^{i,N}(d\mathbf{u})$
"enstrophy ensemble" be collection $\mathcal{E}^{r,N}$ of the $\mu_{En}^{r,N}(d\mathbf{u})$

The proposal is: there should be a 1-1 correspondence between the elements of the two ensembles, ($i.e.$ between ν and En) and:

in corresponding elements the expectation values of **all** local observables will coincide in the limit $N \to \infty$.

Equivalence conjecture ("EC"): *Under equal dissipation condition, i.e. if ν and En verify*

$$\langle W \rangle_{\nu}^{i,N} = \langle W \rangle_{En}^{r,N}$$

it is, for all local observables O:

$$\langle O \rangle_{\nu}^{i} \stackrel{def}{=} \lim_{N\to\infty} \langle O \rangle_{\nu}^{i,N} = \lim_{N\to\infty} \langle O \rangle_{En}^{r,N} \stackrel{def}{=} \langle O \rangle_{En}^{i}$$

for all local observables O. [2, 3, 4, 5]

In the intermittent cases, *i.e.* several attractors $\Rightarrow$ several possible distributions, with the same ν, En, will be distinguished by extra parameters γ, δ. The conjecture is interpreted by extending the 1-1 correspondence to a 1-1 correspondence between the extra labels γ, δ.

Remark: *Strict analogy with the phase transitions in SM*

Parenthesis:

{A **related but different conjecture** deals specifically with the truncated equation, and can be extended to much more general problems: in the special case of cut-off NS the different conjecture can be formulated simply as:

$$\lim_{\nu\to 0} \langle\, O\, \rangle_{\nu}^{i,N} = \lim_{En\to\infty} \langle\, O\, \rangle_{En}^{r,N}$$

at equal power dissipation, as in the previous conjecture, ([6, 7, 8]). The two conjectures are not incompatible: just different. The second could be called fixed number of degrees of freedom e.c. }

Back to the E.C.:

A simple rigorous consequence of EC is:

$$\nu = \lim_{N\to\infty} \mu^{r,N}_{En}(\alpha)$$

this can be a first simple, but demanding, test of the conjecture, *"viscosity test"*.

The α will show very strong fluctuations over the time scale of the largest Lyapunov exponent, at least if the viscosity is so small that a global existence cannot be ascertained for the INS without cut-off.

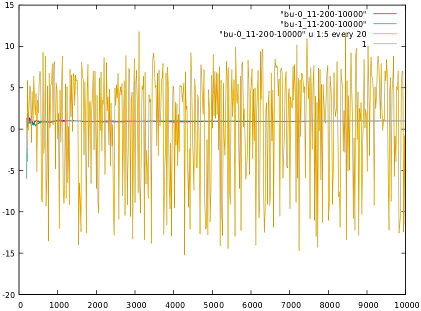
Still the EC implies that the time average of α is ν.

Remark:

Negative values if α “must” show up if ν is small enough **at least for all N large**: otherwise the existence and uniqueness for NS would be implicitly solved, because (it is obvious that) if $\alpha(\mathbf{u}^N(t)) \geq \varepsilon > 0$ for all N large (eventually in t) and for some ε the UV cut-off equation would have N-uniformly smooth solutions **with probability 1 in stationary states**

Tests, only for $d = 2$ and work is in progress for $d = 3$

1) **viscosity test**:



Green: $(t, \frac{1}{t}\int_0^t \frac{1}{\nu}\alpha(\mathbf{u}(\tau))d\tau)$, in RNS (the test)

Red: $(t, \frac{1}{t}\int_0^t \frac{1}{\nu}\alpha(\mathbf{u}(\tau))d\tau)$, in INS, (expected?)

Yellow: value of $(t, \frac{1}{\nu}\alpha(\mathbf{u}(t)))$ in INS, ($\sim$same in RNS)

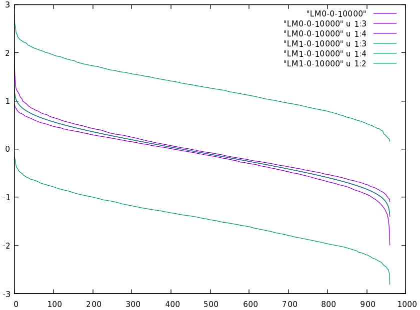
Black: 1

integration step $h = 2^{-17}$, recorded every $4h^{-1}$ steps

time unit 4, total time 10^4 records, $N = 31 \times 31$

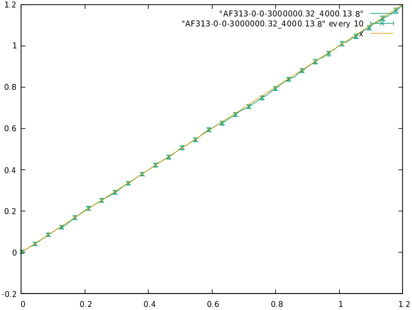
$\nu = 2^{-11} = 2048^{-1}$

2) Lyapunov test:



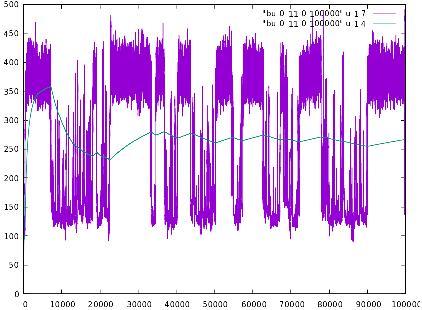
Red: Local Lyapunov exp. $(k, \max_t \lambda_k(t))$ and $(k, \min_t \lambda_k(t))$ for INS
Green: Local Lyapunov exp. $(k, \max_t \lambda_k(t))$ and $(k, \min_t \lambda_k(t))$ for RNS
Blue: common average value of INS and RNS exponents
31×31 resolution. $\nu = 2^{-11}$.
expected??

3) **Reversibility test on INS** (expected ??)



Chaotic hypothesis implies "Fluctuation theorem": if $\sigma(\mathbf{u})$=volume contraction rate with time average σ_+, the $p = \frac{1}{t}\int_0^t \frac{\sigma(\mathbf{u}(\tau))}{\sigma_+} d\tau$ (average "entropy production rate") satisfies large deviation law at rate $s(p)$ s.t. $(s(p) - s(-p)) = p$. Graph is over $3 \cdot 10^6$ INS evolution data **BUT** uses $\sigma(\mathbf{u})$ the reversible observable: *i.e.* irreversible flow looks reversible. 7×7 resolution.

4) Intermittency test



enstrophy as function of t, and its running average, in INS shows intermittency. BUT a check shows that the F.T. does not hold in this case.

There are rather long intervals during which the motion seems to dwell on a single attractor and FT might hold.

15×15-resolution. Problem of precision?

Question:

Is the conjecture $\sim$ correct ?

Remarks

(1): The idea of "equivalent thermostats" goes back, in SM, to [9, 10],

(2): Application to fluids, in a somewhat different form and context, appeared in [11],

(3): In the "weak" form, fixed N and $\nu \to 0$, was discussed in [6, 7, 12, 13, 14, 3, 15, 4] and

(4): In the form discussed here, particularly relevant for fluids, in [3, 4, 5, 16].

(5): One can also consider other ensembles: for instance defining the multiplier $\alpha(\mathbf{u})$ so that energy rather than enstrophy is conserved. The latter ensemble has been considered, in view of the equivalence, in [17] for 3DNS, with remarkable results.

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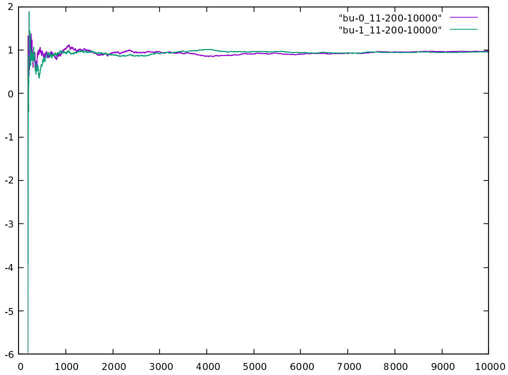
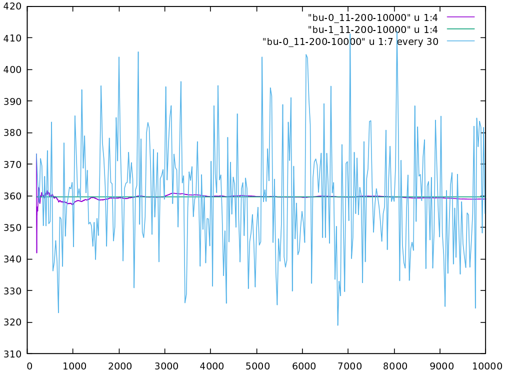
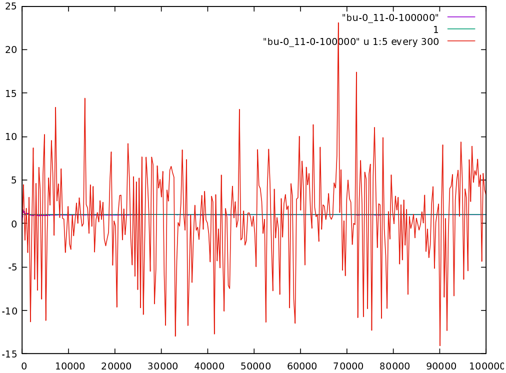


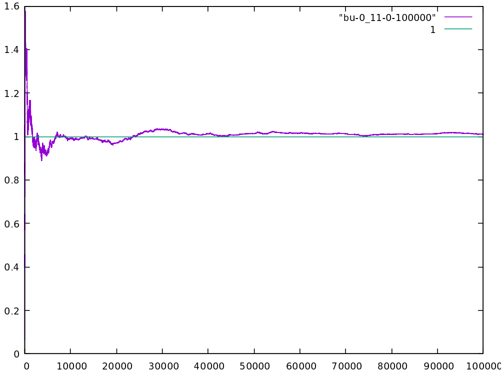
Same viscosity test: only running average of $\frac{\alpha}{\nu}$, 31×31



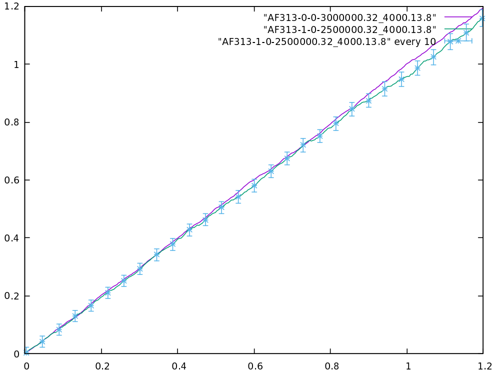
Enstrophy running average and fluctuations in INS and superposed the enstrophy of the equivalent RNS; 31×31



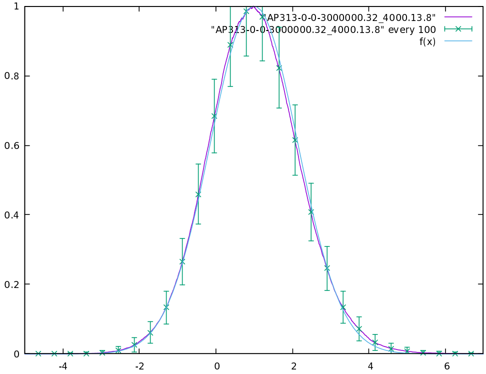
Viscosity test in intermittent flow; 15×15



Same with only running average of $\frac{\alpha}{\nu}$



FT: p-fluctuation for INS and, with error bars, RNS



FT: p-distribution for INS and, with error bars, RNS