Navier-Stokes equation: how relevant the existence-uniqueness problem?

Fluid flows easily reach very large Reynolds numbers and in a container of size L viscosity starts to strongly affecting the flow at the Kolmogorov scale $\ell_K = LR^{-\frac{3}{4}}$.

$$\text{water at } R = 10^4,\ L = 6.m: \qquad l_K = 6.\ mm$$
$$\text{air at } R = 10^{12},\ L = 30.10^3 m: \qquad l_K = 30.\mu m$$

repectively at ∼onset of turbulence or in a ∼cyclone eye of radius 30.km

Here the 'simplest' problem of the properties of a stationary state of an incompressible fluid in a periodic box of side $L = 2\pi$ and subject to a 'large scale' force $\mathbf{f}$ is considered.

The equations of motion for a velocity $\mathbf{u}(\mathbf{x})$ represented as:

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k},c=1,2} u^c_{\mathbf{k}}\, i\, \mathbf{e^c}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}},$$

with $\mathbf{e^c}(\mathbf{k}) = -\mathbf{e^c}(-\mathbf{k})$, $\mathbf{k}\cdot\mathbf{e}^c(\mathbf{k}) = 0$ and $\overline{u}^c_{\mathbf{k}} = u^c_{-\mathbf{k}}$ are:

$$\dot{\mathbf{u}}(\mathbf{x}) = -(\underset{\sim}{\mathbf{u}}(\mathbf{x})\cdot\underset{\sim}{\boldsymbol{\partial}})\mathbf{u}(\mathbf{x}) - \nu\Delta\mathbf{u}(\mathbf{x}) - \boldsymbol{\partial} p(\mathbf{x}) + \mathbf{f}(\mathbf{x})$$

$$\boldsymbol{\partial}\cdot\mathbf{u}(\mathbf{x}) = 0$$

and $\mathbf{f}(\mathbf{x}) = \sum_{|\mathbf{k}|\le k_{\max}} f^c_{\mathbf{k}}\, i\, \mathbf{e^c}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$: hence the constraint $|\mathbf{k}| \le k_{\max}$ indicates that forcing occurs a large scale.

It is 'widely accepted', [**?**], that ℓ_K gives the order of the length scale below which energy, input at large scale, is transfered to be dissipated by the viscosity action.

A theory of NS stationary states should therefore predict at least the averages of observables $O(\mathbf{x})$ whose Fourier's transform $O_{\mathbf{k}}$ vanishes for $|\mathbf{k}| > c\,\ell_K^{-1}$, for some $c = O(1)$.

And, **if** NS equation is considered correct, the latter prediction should concern all local observables, *i.e.* all O's whose Fourier's tranform vanishes except for finitely many harmonics: named local observables.

At this point a difficulty cannot be avoided: given any smooth initial datum $\mathbf{u}_0(\mathbf{x})$ there is no guarantee that there is a solution of the NS $\mathbf{u}_t$ initiating at $\mathbf{u}_0$: *i.e.* **no algorithm exists (so far ?) for constructing $\mathbf{u}_t$**, see [1].

Hence a large part of research has been devoted to properties of *regularized* NS equations: *i.e.* equations modified so that *a priori* it can be guaranteed that the $\mathbf{u}(\mathbf{x})$ evolves remaining smooth and admits an algorithm permitting to construct $(S_t\mathbf{u})(\mathbf{x}) = \mathbf{u}_t(\mathbf{x})$

An example of regularization is to consider the NS eq. as equations for the harmonics $u^c_{\mathbf{k}}$, *i.e.*

$$\dot{u}^c_{\mathbf{k}} = - \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k} \\ a,b=1,2}} T^{a,b,c}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}} u^a_{\mathbf{k}_1} u^b_{\mathbf{k}_2} - \nu \mathbf{k}^2 u^c_{\mathbf{k}} + f^c_{\mathbf{k}}$$

where $T^{a,b,c}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}} \stackrel{def}{=} (\mathbf{e^a}(\mathbf{k_1}) \cdot \mathbf{k_2})(\mathbf{e^b}(\mathbf{k_2}) \cdot \mathbf{e^c}(\mathbf{k}))$.

Then just set $= 0$ all $u^r_{\mathbf{h}}$ with $\mathbf{h} = (h_1, h_2, h_3)$ and $\max_i |h_i| > N$.

This is an ODE, named INS^N (Irreversible NS), on a $D_N = 2((2N+1)^3 - 1)$-dimensional phase space, with only one parameter, **namely** ν, $\mathbf{f}$ is fixed once and for all (with $||\mathbf{f}||_2 = 1$, say, and **on large scale** ($\mathbf{f_k} = 0, |\mathbf{k}| > k_{max}$)).

A rather general property of ODE's generating "chaotic motions" is to admit unique SRB-distributions μ_ν^N, *i.e.* such that, aside from a zero volume set of data $\mathbf{u}$, the averages of *all observables* O are $\mu_\nu^N(O)$, then the stationary properties of the INS^N evolution are completely determined, [2, 3, 4].

Or more generally a finite number of distributions which control the averages of the observables.

To proceed consider viscosities ν (for simplicity) for which there is a unique SRB distribution μ_ν^N for INS^N.

Of course the basic existence problem arisen above has not disappeared: the interest, if the NS equations are taken as fundamental, is entirely resting on the limits as $N \to \infty$ of the local observables averages.

However a similitude with Statistical Mechanics (SM) becomes manifest.

The cut-off N can be seen 'corresponding' to the volume V enclosing the rV hard core particles (say) of a gas of density r not subject to other external forces.

Its (Hamiltonian) eqs. of motion are ODE's that can be seen as a **regularization** of the eqs. that would control motion of an ∞ gas (of density r, filling the Universe !).

SM fared very well in absence of existence-uniqueness results for the evolution of the ∞-system because the physicists' attitude has been:

(1) find, or select, for finite V, a family of stationary distrib. μ^V, and use them to define by $\mu^V(O)$ the averages of physically interesting observ.s (namely the "local" observ.s O, whose value depends only on positions and velocities of particles located in a V*-independent region* inside the confining V, [5]).

(2) show $\lim_{V\to\infty} \mu^V(O) = \langle\, O \,\rangle$ to exist for all "local" O,

(3) **exhibit** general constraints between the average values

In SM item (1) is easy *if the ergodic hypothesis is accepted*, because it allows restricting consideration to stationary μ^V's that are uniform on constant energy surfaces, [6].

Item (2) has been at the center of the study of the "thermodynamic limit", leading to the proofs that, in a very large number of models, the limit exists for *all* local observables, [5, 7].

Item (3) led to the great achievement of showing, in important models, that varying the systems parameters the the averages invariably change in agreement with the variations *forseen by the laws of thermodynamics*, provided V is large enough, [5, 8].

It had also become soon clear, [9], that, besides the uniform distr. on energy surfaces, **other collections of distrib.s could be used to describe the stationary states** of the same system.

The story of SM and the difficulties encountered in the NS equation **may be seen to share common features.**

The volume cut-off V can be considered ıanalogous to the regularization cut-off N for the NS equation.

Role of "local" observables being played in SM by $O(\mathbf{u})$'s depending only on finitely many Fourier modes: *"locality in fluids is locality in Fourier modes while in SM locality is in position space"*. In stationary states of fluids interest is on averages of local observables (*i.e.* on large scale properties of the velocity fields), identified by the distribution μ_ν^N of their values followed over time.

The ergodic hypothesis rests on the chaotic nature of the particles motion: it states that all initial data, *aside from a set of zero volume* in phase space, will yield motions with the same statistical properties (or present a finite number of possibilities, as at phase transitions, [10]) at large V.

The unif. distr, on energy surface can be regarded as a SRB distribution: so in the cahotic evolution of INS^N the **SRB distribution is selected uniquely to describe the statistics of the fluid**. Solving the major problem of identifying the distr. that gives the statistical prop.s of a flow (exceptions allowed as in SM, *e.g.* harmonic chains)

The assumption is inherited from the microscopic motion of the fluid molecules, even in the cases in which the fluid fow is periodic (*e.g.* if viscosity is large) as the fluid equations are derived via scaling limits, without change of the eq.

The just sorted analogy leads to define: *viscosity ensemble* $\equiv$ collection for $\nu > 0$ of SRB stationary distributions μ_ν^N for the INS^N equation.

For each $\nu > 0$ the distr. μ_ν^N assigns the average $\mu_\nu^N(O) = \langle\, O\,\rangle_\nu^N$ of any local observ. O on a flow with initial data randomly selected with a distrib. with a continuous density $\delta(\mathbf{u})$ with respect to the volume in the D_N-dimensional phase space.

As in the corresponding SM case the microcanonical distribution $\mu_E^V(d\mathbf{p}d\mathbf{q})$, for a system of particles of total energy E enclosed in a volume V, assigns the average value $\mu_E^V(O) = \langle\, O\,\rangle_E^V$ to any local observable.

And, still following SM, we call *state with viscosity ν* the distribution μ_ν^N; and *viscosity ensemble* the collection, as ν varies, $\mathcal{E}^N_{viscosity}$ of all the such distributions.

Knowledge of μ_ν^N gives a complete description of the statistical properties of all local observables.

At this point we can ask whether it is possible to define other collections $\mathcal{E}^N$ of stationary distributions $\boldsymbol{\lambda}_{\boldsymbol{\gamma}}^N$ which, depending on a parameter $\boldsymbol{\gamma}$, will assign averages $\langle\, O\,\rangle_\gamma^N =$ so that a **correspondence $\boldsymbol{\nu}\longleftrightarrow\boldsymbol{\gamma}$** can be established in the form $\gamma = g_N(\nu)$ implying:

$$\lim_{N\to\infty}\mu_{\boldsymbol{\nu}}^N(O) = \lim_{N\to\infty}\boldsymbol{\lambda}_{\boldsymbol{\gamma}}^N(O) \qquad \text{if} \quad \boldsymbol{\gamma} = g_N(\nu)$$

Then we shall say that the ensembles $\mathcal{E}^N_{viscosity}$ and $\mathcal{E}^N$ are *equivalent in the $N\to\infty$ limit*.

Just as we say that the microcanonical ensemble distr.s $\boldsymbol{\mu}_E^V$ are equivalent to the canonical ones $\boldsymbol{\lambda}_{\boldsymbol{\beta}}^V$ in the limit as $V \to \infty$ provided $\boldsymbol{\beta}$ and E are suitably related, [7].

Since viscosity describes an average over chaotic microscopic motions it is conceivable that the viscosity could be replaced by another term subject to rapid fluctuations with average $\boldsymbol{\nu}$, while properties of large scale observables (*i.e.* the local ones) will be neglegibly affected.

The possibility of describing the same system with different equations which become equivalent for practical purposes (and even rigorously in suitable limits) for a vast class of observables is familiar in SM: an example is the equivalence between the **microcanonical** and the **isokinetic** ensembles.

In mathematics this would be located in the familiar frame of PDE's *"homogeneization"* phenomena.

Consider a system of $\mathcal{N} = rV$ mass $m = 1$ particles in a cubic vessel V, interacting, with the walls and reciprocally, via repulsive short range potential $\varphi(\mathbf{q})$. The equations:

$$\dot{\mathbf{q}} = \mathbf{p},\ \dot{\mathbf{p}} = -\boldsymbol{\partial}_{\mathbf{q}}\varphi(\mathbf{q}), \qquad \textbf{Hamiltonian or}$$
$$\dot{\mathbf{q}} = \mathbf{p},\ \dot{\mathbf{p}} = -\boldsymbol{\partial}_{\mathbf{q}}\varphi(\mathbf{q}) - \alpha(\mathbf{q},\mathbf{p})\mathbf{p} \qquad \textbf{isokinetic}$$

where the multiplier α is so defined that the second equation admits the total kinetic energy $\frac{1}{2}\mathbf{p}^2$ *exactly* constant. A brief calculation yields the value of α:

$$\alpha(\mathbf{q},\mathbf{p}) = -\frac{\mathbf{p}\cdot\boldsymbol{\partial}_{\mathbf{q}}\varphi(\mathbf{q})}{\mathbf{p}^2}$$

Stationary states of the first eq. are microcanonical distr.s

$$\mu_\varepsilon^V(d\mathbf{p}d\mathbf{q}) = \frac{1}{Z_{\varepsilon,V}}\,\delta(\frac{\mathbf{p}^2}{2} + \varphi(\mathbf{q}) - \varepsilon\mathcal{N})\,d\mathbf{p}d\mathbf{q}$$

while stationary distributions for the second are:

$$\lambda_\beta^V(d\mathbf{p}d\mathbf{q}) = Z_{\beta,V}^{-1}\delta(\frac{\mathbf{p}^2}{2} - \frac{3}{2}\beta^{-1}\mathcal{N})d\mathbf{p}d\mathbf{q}$$

The first is well known, while the second can be checked directly, [11]. Then it can be shown, in absence of phase transitions, that local observables O verify

$$\lim_{V\to\infty}\mu_\varepsilon^V(O) = \lim_{V\to\infty}\lambda_\beta^V(O)$$

under the "equivalence condition" $\mu_\varepsilon^{\mathbf{V}}(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{p^2}) = \frac{\mathbf{3}}{\mathbf{2}}\beta^{-\mathbf{1}}\mathcal{N}$.

Since 1980's different equations are used describing the same system and yielding same averages to interesting observables (at least approximately: complete equivalence could only be in limit situations, not really accessible).

Vast literature on numerical simulations on nonequilibrium, [11, 12, 13] provides many examples.

Different equations for the same system were usually obtained by adding to the equations new terms so designed to turn one, or more, selected (typically non-local) observable into a constant of motion.

The extra terms have been often interpreted as simulating the action of "thermostats": such are the "Nosè-Hoover" thermostats, [14], or the "Gaussian" thermostats, [15]. It is even possible to impose simultaneously many extra terms: a most remarkable case in [16].

In general the selection of the observables which, via the modification of equations, must remain constant is addressed towards quantities that are expected to have small fluctuations in a limit situation of interest, like the total kinetic energy in the above example.

For instance in the case of [16] the NS equation is replaced by 'thermostats' (*i.e.* extra terms) imposing that the energy of several shells above the Kolmogorov length scale (where friction is believed to have little effect) is constrained to keep the value predicted by the Kolmogorov law, [16].

Coming back to the NS equations

$$\dot{u}^c_{\mathbf{k}} = -\sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}\\ a,b=1,2}} T^{a,b,c}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}} u^a_{\mathbf{k}_1} u^b_{\mathbf{k}_2} - \nu \mathbf{k}^2 u^c_{\mathbf{k}} + f^c_{\mathbf{k}}$$

It has been proposed, [17, 18, 19], to change the viscosity ν into a multiplier α so defined that the resulting evolution keeps the **enstrophy** $\mathcal{D}(\mathbf{u}) = \sum_{\mathbf{k}} \mathbf{k}^2 \mathbf{u}^2_{\mathbf{k}}$ constant. This is obtained by defining:

$$\alpha(\mathbf{u}) = \frac{\sum_c \sum_{\mathbf{k}} (-\mathbf{n}^c_{\mathbf{k}}(\mathbf{u}) \mathbf{k}^2 \overline{u}^c_{\mathbf{k}} + \mathbf{k}^2 f^c_{\mathbf{k}} \overline{u}^c_{\mathbf{k}})}{\sum_c \sum_{\mathbf{k}} \mathbf{k}^4 |u^c_{\mathbf{k}}|^2} \qquad (*:\ RNS^N)$$

where $\mathbf{n}^c_{\mathbf{k}}(\mathbf{u})$ is the non-linear term in the eq.

The R in the name RNS stands to stress that the equation RNS^N is time reversible, unlike the irreversible INS^N.

Or it is possible to fix α so that the **energy** $\mathcal{E}(\mathbf{u}) = \sum_c \sum_{\mathbf{k}} |u^c_{\mathbf{k}}|^2$ is exactly constant, which leads to a much simpler multiplier α:

$$\alpha(\mathbf{u}) = \frac{\sum_c \sum_{\mathbf{k}} f^c_{\mathbf{k}} \overline{u}^c_{\mathbf{k}}}{\sum_c \sum_{\mathbf{k}} \mathbf{k}^2 |u^c_{\mathbf{k}}|^2} \qquad (** : \ ENS^N))$$

studied in [20].

Physical interpr.: thermostats are forces with the effect of removing heat generated by the forcing. In the case of an incompressible fluid above heat has to be taken away (in either the enstrophy thermostat or in the energy thermostat) to maintain the relation between pressure and temperature at constant density prescribed by th equation of state.

The stationary distributions for the equation, referred as RNS^N, with α in (*) are parameterized by the enstrophy value D as $\lambda_{\mathbf{D}}^{\mathbf{N}}$ and their collection will be called **"enstrophy ensemble"**, $\mathcal{E}^N_{enstrophy}$.

Likewise the stationary distributions for the equation, referred as ENS^N, with α in (**) are parameterized by the value E as $\theta_{\mathbf{E}}^{\mathbf{N}}$ and form the **"energy ensemble"** $\mathcal{E}^N_{energy}$.

Given viscosity ν suppose, for simplicity, that there is only one stationary distribution $\mu^N_\nu \in \mathcal{E}^N_{viscosity}$ for all N large:

Conjecture: *Let $D = \mu^N_\nu(\mathcal{D})$ be the average enstrophy. Then also the distribution $\lambda^N_D \in \mathcal{E}^N_{enstrophy}$ is unique. The distributions μ^N_ν, λ^N_D are equivalent in the sense*

$$\lim_{N\to\infty} \mu^N_\nu(O) = \lim_{N\to\infty} \lambda^N_D(O) \qquad (@)$$

In other words the viscosity ensemble and the enstrophy ensembles are equivalent in the limit $N \to \infty$ provided their entrophies agree, if the stationary distr. is unique.

More generally the conjecture is interpreted as saying that the SRB distributions for the INS^N equation can be put in one-to-one correspondence with the distributions for the RNS^N equation with the same enstrophy so that for corresponding distributions @ holds.

For a first test remark a non trivial consequence: namely if both sides of INS^N or $RNS^N$are multipled by $\overline{u}^c_{\mathbf{k}}$ and summed over $c, \mathbf{k}$ one finds, respectively:

$$\frac{d}{dt}\mathcal{E}(\mathbf{u}) = -\nu\mathcal{D}(\mathbf{u}) + \mathbf{f}\cdot\mathbf{u}, \qquad \frac{d}{dt}\mathcal{E}(\mathbf{u}) = -\alpha(\mathbf{u})D + \mathbf{f}\cdot\mathbf{u}$$

(no non-linear terms: energy conservation for $\mathbf{f} = \vec{0}, \nu = 0$).

Since the equivalence condition is $\langle\,\mathcal{D}\,\rangle_\nu^N = D$ and $O = \mathbf{f}\cdot\mathbf{u}$ is a local observable it follows that the averages $\langle\,\mathbf{f}\cdot\mathbf{u}\,\rangle$ must be equal in the limit $N\to\infty$ and therefore

$$\nu = \lim_{N\to\infty}\langle\,\alpha\,\rangle_D^N$$

because the averages of $\frac{d}{dt}\mathcal{E}(\mathbf{u})$ must vanish.

There are already a few numerical tests of the equivalence of the ensembles INS^N and RNS^N for the 2D fluid evolution, [17, 18, 18], and for the 3D fluid [21, 19].

For the ENS^N tests relevant for the conjecture have been proposed in [20].

1) The simple test of equivalence $\langle\,\alpha\,\rangle_D^N \xrightarrow[N\to\infty]{} \nu$ has been performed in 2D and 3D: with positive results in all published cases: see Fig.4 in [17] and Fig.1 in [22], Fig.4 in [21], Fig.15a in [19].

2) The 2D tests have shown that in many cases equivalence holds also for observables that are non local. Remarkable is the observable $\alpha(\mathbf{u})$ studied as an observable for the INS^N equation. It turned out, with an exception in Fig.4 of [21], that it also averages to ν while presenting smaller fluctuations compared to the RNS^N case, see [17] for 2D case and Fig.16a in [19] for 3D;

3) The latter remark led to tests equivalence of other typically non local observables). A few tests, only in 2D so far, have been performed comparing, under the equivalence condition, the spectra of the symmetric part $J(\mathbf{u})$ of the $D_N \times D_N$ Jacobian matrix $\frac{\partial \dot u^c_{\mathbf{k}}}{\partial \dot u^b_{\mathbf{h}}} \stackrel{def}{=} J(\mathbf{u})_{c,\mathbf{k};b,\mathbf{h}}$.

Such observables are related to the Lyapunov exponents, [23, 24]. The result has been that essentially the eigenvalues *averaged over the flows* agree if ordered in the same way (*e.g.* in decreasing order): see Fig.7 in [22] and Fig.5 in [17].

Most remarkable is that, while the average of the eigenvalues agree surprisingly well, the eigenvalues of the $J(\mathbf{u})$ reach equal averages, along the two evolutions, in spite of much larger fluctuations in the RNS^N evolution compared to the INS^N, see Fig.6 in [22].

4) The 3D tests are still **somewhat preliminary**: yet they yield important informations. If the conjecture is correct it is expected that in RNS^N the fluctuating viscosity α fluctuates considerably and events in which $\alpha < 0$ occur.

The reason is that **otherwise** it can be proved that D being bounded ($\nu\langle\,\mathcal{D}\,\rangle = \varepsilon$ is expected bounded, fixed $\mathbf{f}$) it would follow that the velocity $\mathbf{u}$ remains smooth with all derivatives bounded uniformly in N, see [19, Appendix], thus giving a new prespective to the question of existence and regularity of the NS flows.

It is therefore surprising, at least if ν is so small that the fluid is certainly in a turbulent regime, that for N large velocity fields $\mathbf{u}(t)$ with $\alpha(\mathbf{u}(t)) < 0$ are **not observed** (after a short transient time depending on the initial data) in several experiments, [21, 19].

Question to be understood is whether events with $\alpha < 0$ are not seen because they are rare events (which is my expectation), so rare to be missed (when N is large) in time series with too large time step and/or integration step. For some evidence on this phenomenon, see Fig.15 in [19].

5) The results in [19] suggest that the conjecture above **is too strong and might fail** unless the definition of local observable is *deeply modified* restricting the notion of local observable, for the purpose of the conjecture.

As formulated above the only requirement for locality is that O depends only on a finite number of harmonics $\mathbf{u}_\mathbf{k}$: hence it would be possible to claim equivalence for an observable which depends on one Fourier component with $|\mathbf{k}| > k_\nu$ *if N is large enough*.

But from [19] it **emerges that equivalence is not verified in several such tests**: a further condition appears needed, *i.e.* that O depends only on the components $\mathbf{u}_\mathbf{k}$ with $|\mathbf{k}| < c_0 k_\nu$ for some constant c_0 of order 1. See "Conjecture 2" and Fig.11–13 in [19] which suggest a value $c_0 \sim \frac{1}{8}$. The evidence is not yet conclusive, in my view, and needs more data to really exclude $c_0 = \infty$.

6) Tests of the possibility of existence of several attractors have shown that even in presence of Chaotic motions there are cases in which multiple attractors can coexist showing strong intermittency phenomena, see figure below.

7) The remark 4) suggests that the approach to the theory of the NS equation based on searching for existence and uniqueness in function spaces could be usefully extended to equivalent equations: while not solving nor simplifying the problem it can open new perspectives, just like introducing new equilibrium ensembles does not solve basic mathematical problems of SM but, actually, introduces new ones overcompensated by the deeper understanding of thermodynamics.

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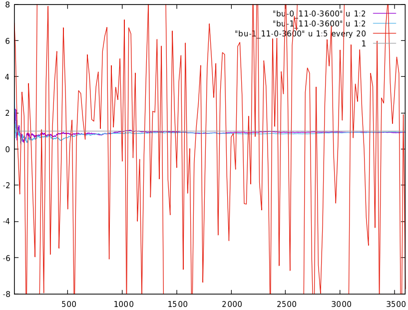

Fig.4 [17] 2D: $R = 2048$, $N = 15$, 960 modes, $h = 2^{-17}$

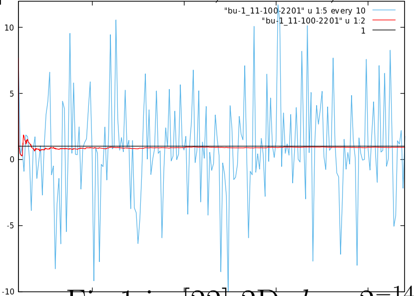

Fig.1 in [22] 2D: $h = 2^{-14}$, $R = 2048$, $N = 31$, 3968 modes

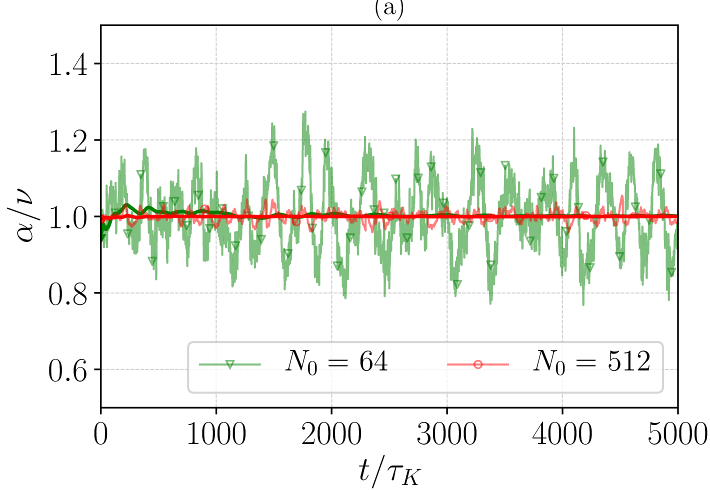

Fig.15a in [19] 3D $\frac{\alpha}{\nu}$ and $\alpha > 0$??

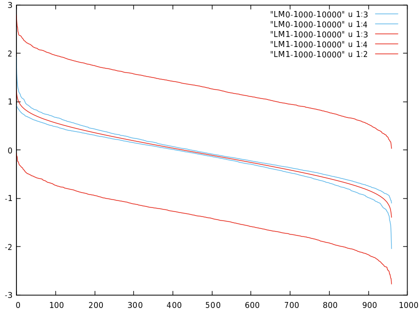

Fig.6 in [17] red=max,min $\overline{\lambda}_k$ in RNS – green=max,min in INS, average in BOTH cases= central red

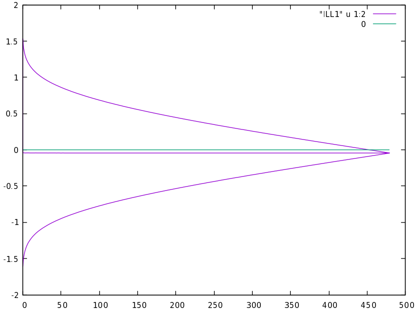
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Fig.7 [22]