Statistical ensembles in fluid dynamics

Incompressible NS equations on $[0,2\pi]^d$ with periodic b.c., large scale forcing $\mathbf{f}$, UV cut-off N (eventually $\to\infty$) for $\mathbf{u}(\mathbf{x})=\sum_{c=1}^{d-1}\sum_{|\mathbf{k}|\le N} i\,\mathbf{e}^c(\mathbf{k})\,u^c_{\mathbf{k}}\,e^{i\mathbf{k}\cdot\mathbf{x}}$ are:

$$\dot u^c_{\mathbf{k}}=\mathbf{n}(\mathbf{u},\mathbf{u})^c_{\mathbf{k}}-\nu\mathbf{k}^2u^c_{\mathbf{k}}+\mathbf{f}_{\mathbf{k}},\qquad \text{“INS}^{\text{N}}\text{ eqs.”}$$

($\mathbf{f}_{\mathbf{k}}=0$ for $|\mathbf{k}|>\overline{k}$ fixed all over, $||\mathbf{f}||=1$.)

Call μ^N_ν the stationary prob. distributions (SRB).

Their collection $\mathcal{E}^N_{viscosity}$ defines “viscosity ensemble”;

Concentrate attention on the *local observables*: *i.e.* functions $O(\mathbf{u})$ depending only on finitely many $u^c_{\mathbf{k}}$.

The random nature of the viscosity ν suggests that it can be replaced by an observable that fluctuates chaotically. Generating a new evolution but equivalent to it.

Thus the following equation is proposed to generate an ensemble $\mathcal{E}'$ equivalent to $\mathcal{E}^N_{viscosity}$:

$$\dot{u}^c_{\mathbf{k}} = \mathbf{n}(\mathbf{u},\mathbf{u})^c_{\mathbf{k}} - \boldsymbol{\alpha}(\mathbf{u})\,\mathbf{k}^2 u^c_{\mathbf{k}} + \mathbf{f}_{\mathbf{k}}, \qquad \text{“RNS}^{\text{N}}\text{ eqs.”}$$

where $\boldsymbol{\alpha}$ is such that $\mathcal{D} = \sum_{c,|\mathbf{k}|\le N} \mathbf{k}^2 |u^c_{\mathbf{k}}|^2$, **enstrophy**, is an exact constant. $\boldsymbol{\alpha}(\mathbf{u})$ value is (by inspection)

$$\boldsymbol{\alpha}(\mathbf{u}) = \frac{\sum_{\mathbf{k},c}\Big(\mathbf{n}(\mathbf{u},\mathbf{u})^c_{\mathbf{k}}\mathbf{k}^2\overline{u}^c_{\mathbf{k}} + \mathbf{f}^c_{\mathbf{k}}\mathbf{k}^2\overline{u}^c_{\mathbf{k}}\Big)}{\sum_{\mathbf{k},c}\mathbf{k}^4|u^c_{\mathbf{k}}|^2}$$

The RNS$^{\text{N}}$ eqs. stationary distr.s will be parameterized by the value of the enstrophy D and denoted ρ^N_D.

Their collection forms **“enstrophy ensemble”**, $\mathcal{E}_{enstrophy}$

RNS$^{\text{N}}$ eqs is *reversible* and call α “reversible viscosity”.

Recall enstrophy definition $\left[\mathcal{D}(\mathbf{u}) \stackrel{def}{=} \sum \mathbf{k}^2 |u^c_{\mathbf{k}}|^2\right]$

Conjecture: *Let* $\mu^N_\nu(O)$, $\rho^N_D(O)$, **O=local observable**.
If D **is related to** ν **(and** N**) by:**

$$D = \mu^N_\nu(\mathcal{D})$$

then **for all local observables** O *it will be*

$$\lim_{\mathbf{N}\to\infty} \mu^{\mathbf{N}}_\nu(\mathbf{O}) = \lim_{\mathbf{N}\to\infty} \rho^{\mathbf{N}}_{\mathbf{D}}(\mathbf{O}),$$

"INS and RNS equations are equivalent", on local obs., and on condition of equal enstrophy once UV cut of is removed.

Strict analogy with ensembles equivalence in SM: *e.g.* microcanonical ensemble and isokinetic ensemble are equivalent on condition of equal average kinetic energy (in Hamilton eqs.) and equal kinetic energy (in isokinetic eqs.) in the limit $V \to \infty$ on local observables).

A rigorous consequence: equivalence condition implies

$$\lim_{N\to\infty}\rho_D^N(\alpha)=\nu, \qquad \Big[\leftarrow\ W=\mathbf{F}\cdot\mathbf{u} \text{ is local O}\Big]$$

So corresponding distributions equivalence implies "reversible viscosity" α has average ν: "*homogeneization*".

Homogeneisation tests by various groups.
In **2D** up to $\sim 32^2\sim \mathbf{10^3}$ harmonics $\mathbf{u}_\mathbf{k}$, $\nu\sim \mathbf{10^{-3}}$
In **3D** up to $\sim 340^3\sim \mathbf{5\cdot 10^7}$ in $d=3$ and ν in $(10^{-1},10^{-5})$.

In **3D** conjecture tests **on several local $O(\mathbf{u})$'s** confirm it but in a weaker form: it has been necessary to **restrict to** $O(\mathbf{u})$ localized on scales larger than a constant $\mathbf{c}$ of order 1 times the Kolmogorov scale

$$K_{kolmogorov}=\Big(\frac{D}{\nu^2}\Big)^{\frac14}=\Big(\frac{\eta}{\nu^3}\Big)^{\frac14}$$

i.e. $O(\mathbf{u})$'s **only functions of $u_\mathbf{k}$ with $|\mathbf{k}|<\mathbf{cK_{kolmogorov}}$**.

In my opinion a firm conclusion on $c < +\infty$ requires further simulations (larger N, finer integration step ..).

Open question/test is whether in the reversible evolution the multiplier α (with average ν) has a distribution which gives probability zero to $\mathbf{u}$'s with $\alpha(\mathbf{u}) \leq 0$.

Important: if on the attractor $\alpha(\mathbf{u}) > \varepsilon > 0$ then enstrophy constancy $\Rightarrow$ **attractor consists of ∞-smooth velocity fields**.

So either α fluctuates below 0 or conjecture is likely false. From the simulations it seems that events with $\alpha < 0$ are possible even though very rare.

Conclusion: perhaps setting NS equations in a Sobolev space is not the only physically sensible option.

See Appended References for

1) **A first equivalence example:** [1]
2) **Path to the conjecture:** [2, 3, 4, 5]
3) **3D enstrophy ensemble:** [5, 6]
4) **3D energy ensemble:** [7]
5) **Shell model:** [8]
7) **Stat-Mech:** [9, 10, 11, 12, 13]
8) **Turbulence physics:** [14, 15, 16, 17, 18]

See Appended Simulations Examples

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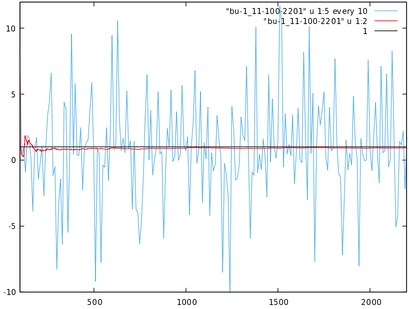

Fig.1 in [19] **2D**: $h = 2^{-14}$, $R = 2048$, $N = 31$, 3968 modes. Data of $\frac{\alpha(\mathbf{u}(t)}{\nu}$ (blue fluctuations) are registered at multiples of h^{-1} by 4: the plot looks at such data and interpolates by lines every 10 of them (to avoid seeing just a stain). The read line is the running averages of the $\frac{\alpha(\mathbf{u}(t)}{\nu}$: $\frac{1}{t}\int_0^t \frac{\alpha(\mathbf{u}(t'))}{\nu} dt'$ which by the conjecture should tend to 1 represented by the black horizontal line.

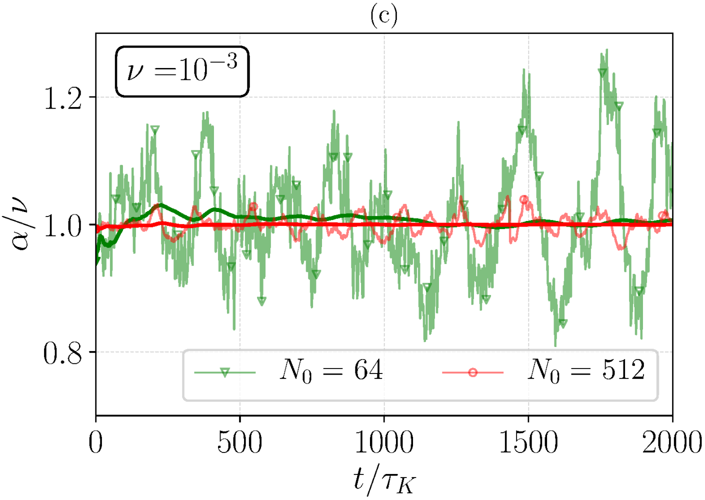

Fig.16c in [5]: **3D** $\frac{\alpha}{\nu}$ and $\boldsymbol{\alpha} > 0$???, ($N = 21$ (*i.e.* $\sim 8 \cdot 10^4$ modes)) or and $N = 170$ (*i.e.* $\sim 4 \cdot 10^7$ modes), $R = 10^3$. Remark that the fluctuations are smaller at large N.

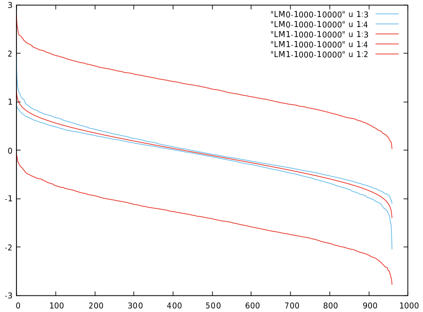

Fig.6 in [4]: $R = 2048$, 920 harmonics ($N = 15$). Plot of Local Lyap. exp. (*i.e.* spectrum of the Jacobian matrices $J(\mathbf{u}(t))$, *i.e.* $\lambda_k(\mathbf{u})$, $k = 0, \ldots, 920$, **2D**)

red= upper and lower lines, $\max_t, \min_t\ \lambda_k(t)$ in **RNS**

blue=$\max_t, \min_t\ \lambda_k(t)$ in **INS**,

average: in **BOTH** cases $\overline{\lambda}_k$ averaged for each k over in the **central red** line

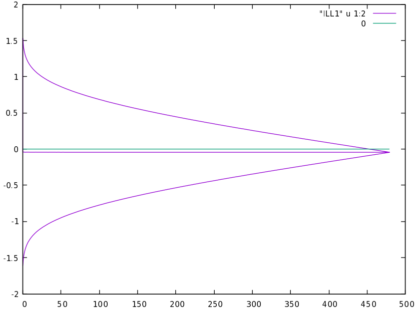

Fig.7 in [19]: **Quasi pairing of Lyapunov's** RNS & INS $\mathcal{N} = 920$ harmonics of the previous figure.

The $\overline{\lambda}_k$, average local spectrum of $J(\mathbf{u}(t))$ (central red line, equal for RNS and INS) of the previous figure, are plotted as $(k, (\overline{\lambda}_k + \overline{\lambda}_{\mathcal{N}-1-k}))$, $k = 0, \ldots, \mathcal{N} - 1$, showing approximately a "pairing" to a level < 0 (equal to $\sim \frac{2}{\mathcal{N}}$ times the average phase space contraction $\overline{div} = \sum_k \overline{\lambda}_k$). However this is likely due to the small N ($N = 15$ in this case): for larger N the graph of $(\overline{\lambda}_k + \overline{\lambda}_{\mathcal{N}-1-k})$ is expected to be a decreasing curve.