Navier-Stokes equation: how relevant the existence-uniqueness problem?

Abstract: Existence-uniqueness theorems may be too strict requirements for many problems in Physics: Statistical Mechanics flourishes studying systems for which no existence-uniqueness is available for most infinite systems to which ideally it should apply in studying thermodynamics. Here an analogy is proposed between the theory of the thermodynamic limit and the problem of fluids and turbulence discussing pro-and-con for a statistical interpretation of viscosity and reversibility of fluid motion, also with attention to recent computer simulations.

*

Navier-Stokes equation: how relevant the existence-uniqueness problem?

Properties of a stationary state of an incompressible fluid in a periodic box of side $L = 2\pi$ and subject to a 'large scale' force f is considered.

'widely accepted', [1]: $\ell_K = LR^{-\frac{3}{4}}$ gives order of length-scale below which energy, input at large scale, is transferred to be dissipated by the viscosity action.

A stationary NS-states theory should predict at least averages of observables $O(\mathbf{u})$ depending on u via large scale Fourier's components $\mathbf{u_k}$, *i.e.* with $|\mathbf{k}| < c\,\ell_K^{-1}$, for some $c = O(1)$.

Represent a velocity $\mathbf{u}(\mathbf{x})$ as:

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}\neq\mathbf{0},c=1,2} u^c_{\mathbf{k}}\, i\, \mathbf{e^c}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}, \qquad \mathbf{x} \in [-\pi,\pi]^3$$

$||\mathbf{e^c}(\mathbf{k})|| = \mathbf{1}$, two unit vectors

$\mathbf{k}\cdot\mathbf{e}^c(\mathbf{k}) = 0$,

$\mathbf{e^1}(\mathbf{k})\cdot\mathbf{e^2}(\mathbf{k}) = 0$ and $\mathbf{e^c}(\mathbf{k}) = -\mathbf{e^c}(-\mathbf{k})$,

$\overline{u}^c_{\mathbf{k}} = u^c_{-\mathbf{k}}$ complex scalars.

The Navier-Stokes equations of motion are:

$$\dot{\mathbf{u}}(\mathbf{x}) = -(\underset{\sim}{\mathbf{u}}(\mathbf{x}) \cdot \underset{\sim}{\boldsymbol{\partial}})\mathbf{u}(\mathbf{x}) - \nu\Delta\mathbf{u}(\mathbf{x}) - \boldsymbol{\partial}p(\mathbf{x}) + \mathbf{f}(\mathbf{x})$$

$$\boldsymbol{\partial} \cdot \mathbf{u}(\mathbf{x}) = 0$$

and $\mathbf{f}(\mathbf{x}) = \sum_{0<|\mathbf{k}|\le k_{\max}} f^c_{\mathbf{k}}\, i\, \mathbf{e}^{\mathbf{c}}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$**:**

constraint $|\mathbf{k}| \le k_{\max}$ **indicates that forcing occurs at large scale.**

Observables $O(\mathbf{u})$**'s depending on finitely many harmonics of u will be at center of attention here; they will be called** *i.e.* **"large scale observables" or**

"LOCAL OBSERVABLES"

If NS is considered fundamentally correct, it should predict properties of the time averages of all local observables, not just those of scale above ℓ_K.

Unavoidable difficulty: no guarantee for a NS-solution, $S_t\mathbf{u}_0 \stackrel{def}{=} \mathbf{u}_t$, initiating at a smooth $\mathbf{u}_0$: *i.e.* **no stable algorithm exists for constructing $\mathbf{u}_t$**, see [2].

Research mostly devoted to *regularized* **NS eq.:** *i.e.* **modified so that** *a priori* **$\mathbf{u}(\mathbf{x})$ evolves remaining smooth and admits an algorithm to construct $(S_t\mathbf{u})(\mathbf{x}) = \mathbf{u}_t(\mathbf{x})$**

Example of regularization:
write NS as equations for the harmonics $u^c_{\mathbf{k}}$, *i.e.*

$$\dot u^c_{\mathbf{k}} = -\sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}\\ a,b=1,2}} T^{a,b,c}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}} u^a_{\mathbf{k}_1} u^b_{\mathbf{k}_2} - \nu \mathbf{k}^2 u^c_{\mathbf{k}} + f^c_{\mathbf{k}}$$

where $T^{a,b,c}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}} \stackrel{def}{=} (e^a(\mathbf{k}_1)\cdot \mathbf{k}_2)(e^b(\mathbf{k}_2)\cdot e^c(\mathbf{k}))$.

Set $= 0$ *all* $u^r_{\mathbf{h}}$ *with* $\mathbf{h} = (h_1, h_2, h_3)$ *and* $\max_i |h_i| > N$.

This ODE, named $\mathrm{INS}^{\mathrm{N}}$ **(Irreversible NS), is on a** $D_N = 2((2N+1)^3 - 1)$**-dimensional phase space, f is fixed once and for all (with** $||\mathbf{f}||_2 = 1$**, say, and on large scale (**$\mathbf{f}_{\mathbf{k}} = 0, |\mathbf{k}| > k_{max}$**)): depends on** $\boldsymbol{\nu}$**, only**

A general property of ODE's generating "chaotic motions", like the $\mathrm{INS}^{\mathrm{N}}$**, is very important: it is to admit, generically, an unique SRB-distribution** μ^N_ν**.**

This means (Ruelle) that, aside from a zero volume set of data u, time averages of *all observables* O are obtained by integration $\langle O \rangle = \boldsymbol{\mu}^N_{\boldsymbol{\nu}}(O)$.

Hence stationary properties of the $\mathrm{INS^N}$ evolution (which is certainly chaotic at small $\boldsymbol{\nu}$) are completely determined by SRB, [3, 4, 5].

At $\boldsymbol{\nu}$'s where uniqueness of the SRB distr. can be assumed, this answers "which is the probability distr. relevant for the averages" among the uncountably many stationary ones ? :

it is the SRB distr. $\boldsymbol{\mu}^N_{\boldsymbol{\nu}}$.

Of course the basic existence problem has not disappeared: forgetting that $\boldsymbol{\mu_\nu^N}$ is not known, the interest, if the NS equations are taken as fundamental, is entirely resting on the limits as $N \to \infty$ of the local observables averages.

But analogy with Stat. Mechanics (SM) is manifest.

(Hamiltonian) eqs. of motion for rV hard core particles (say) in a volume V can be seen as ODE regularizing the evolution of an infinite gas of density r for which no constructive existence - uniqueness is known.

Still SM fared very well in absence of existence-uniqueness results for the evolution of the ∞-system, because of the physicists' attitude.

The ergodic hypothesis can be interpreted as identifying the microcanonical distributions $\boldsymbol{\mu}_E^V$ with the SRB distributions for the chaotic microscopic evolution described by the finite volume Hamiltonian ODE.

Thus in SM the time average of the value of an observable $O(\mathbf{p},\mathbf{q})$ is simply $\langle\, O\,\rangle_E^V \equiv \boldsymbol{\mu}_E^V(O)$: the O's are restricted to be LOCAL

i.e. **as V varies their values depend only on positions-velocities of particles located in a** *V-independent region* **inside the confining V, [6].**

In SM it is then shown that (in many models) $\lim_{V\to\infty}\boldsymbol{\mu}^V(O) = \langle\, O\,\rangle$ to exist $\forall$ "local" O: "thermodynamic limit"b

And constraints between the various average values are exhibited to lead to the great achievement of showing, in important models, that varying the systems parameters the averages invariably change in agreement with the variations *forseen by the laws of thermodynamics*, **provided V is large enough, [6, 7].**

Analogy with fluids

Unif. distr. on energy surface $\longleftrightarrow$ SRB distribution. In $\mathrm{INS}^{\mathrm{N}}$ (chaotic) the SRB uniquely describes the stationary statistical properties.

The assumption is inherited from the microscopic motion of the fluid molecules, *even* when the fluid flow is periodic (*e.g.* if viscosity is large): as the fluid equations are derived via scaling limits, without change of the eq. of motion.

The cut-off N in INS^N plays the role of the finite volume cut-off V in SM; both look at average properties of a restricted class of observables: *i.e.* the local ones.

The just sorted analogy leads to define: *viscosity ensemble* $\equiv$ **collection for** $\boldsymbol{\nu} > 0$ **of SRB stationary distributions** $\boldsymbol{\mu}_{\boldsymbol{\nu}}^{N}$ **for the** $\mathrm{INS}^{\mathrm{N}}$ **equation.**

For each $\boldsymbol{\nu} > 0$ **the distr.** $\boldsymbol{\mu}_{\boldsymbol{\nu}}^{N}$ **assigns the average** $\boldsymbol{\mu}_{\boldsymbol{\nu}}^{N}(O) = \langle\, O\,\rangle_{\nu}^{N}$ **of any local observ.** O **on a flow with initial data randomly selected with a distrib. with a density with respect to the volume** in the D_N-dimensional phase space.

In the corresponding SM case the microcanonical distribution $\boldsymbol{\mu}_{E}^{V}(d\mathbf{p}d\mathbf{q})$**, for a system of particles of total energy** E **enclosed in a volume** V**, assigns the average value** $\boldsymbol{\mu}_{E}^{V}(O) = \langle\, O\,\rangle_{E}^{V}$ **to any local observable.**

At this point we ask whether it is possible to define other collections $\mathcal{E}^N$ of stationary distributions $\boldsymbol{\lambda}_{\boldsymbol{\gamma}}^N$ which, depending on a parameter $\boldsymbol{\gamma}$, will assign averages $\langle O \rangle_{\gamma}^N =$ so that a correspondence $\boldsymbol{\nu} \longleftrightarrow \boldsymbol{\gamma}$ can be established in the form $\boldsymbol{\gamma} = g_N(\boldsymbol{\nu})$ implying:

$$\lim_{\mathbf{N}\to\infty} \mu_{\boldsymbol{\nu}}^{\mathbf{N}}(\mathbf{O}) = \lim_{\mathbf{N}\to\infty} \boldsymbol{\lambda}_{\boldsymbol{\gamma}}^{\mathbf{N}}(\mathbf{O}) \qquad \text{if} \quad \boldsymbol{\gamma} = g_N(\boldsymbol{\nu})$$

Then we shall say that the ensembles $\mathcal{E}_{viscosity}^N$ and $\mathcal{E}^N$ are *equivalent* (in the $N \to \infty$ limit).

Just as we call microcanonical distr.s $\boldsymbol{\mu}_E^V$ equivalent to canonical ones $\boldsymbol{\lambda}_{\boldsymbol{\beta}}^V$ in the limit as $V \to \infty$ provided β and E are suitably related. [8].

Viscosity phenomenologically describes an average over chaotic microscopic motions it is conceivable that it could be replaced by another force subject to rapid fluctuations with average ν, while properties of large scale observables (*i.e.* the local ones) will be neglegibly affected.

Describing the same system with different equations which become equivalent for practical purposes (and even rigorously in suitable limits) for a vast class of observables is familiar in SM: an example is the equivalence between the microcanonical and the isokinetic ensembles.

Since 1980's different equations are used describing the same system and yielding same averages to interesting observables (at least approximately: complete equivalence could only be in limit situations, like $V \to \infty$, not really accessible).

Vast literature on simulations on nonequilibrium, [9, 10, 11] provides many examples.

Different equations for the same system: usually obtained by adding to the equations new forces so designed to turn one, or more, selected (typically non-local) observable into a constant of motion.

The extra forces have been often interpreted as simulating the action of "thermostats": such are the "Nosè-Hoover" thermostats, [12], or the "Gaussian" thermostats, [13]. It is even possible to impose simultaneously many extra forces: a most remarkable case in [14] concerns the NS eq..

Selection of the observables which, via the modification of equations, must remain constant is addressed towards quantities that are expected to have small fluctuations in a limit situation of interest, like kinetic energy in the example.

Coming back to the NS equations

$$\dot{u}^c_{\mathbf{k}} = -\sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}\\ a,b=1,2}} T^{a,b,c}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}} u^a_{\mathbf{k}_1} u^b_{\mathbf{k}_2} - \boldsymbol{\nu}\mathbf{k}^2 u^c_{\mathbf{k}} + f^c_{\mathbf{k}} \qquad (\#)$$

It is proposed, [15, 16, 17], to change the viscosity $\boldsymbol{\nu}$ into a multiplier $\boldsymbol{\alpha}$ so defined that the evolution keeps constant "enstrophy" $\mathcal{D}(\mathbf{u}) \stackrel{def}{=} \sum_{\mathbf{k}} \mathbf{k}^2 \mathbf{u}^2_{\mathbf{k}}$ ("enstrophy thermostat"). Achieved by:

$$\boldsymbol{\alpha}(\mathbf{u}) = \frac{\sum_{\mathbf{c}}\sum_{\mathbf{k}}(-\mathbf{t}^{\mathbf{c}}_{\mathbf{k}}(\mathbf{u})\mathbf{k}^2\overline{\mathbf{u}}^{\mathbf{c}}_{\mathbf{k}} + \mathbf{k}^2\mathbf{f}^{\mathbf{c}}_{\mathbf{k}}\overline{\mathbf{u}}^{\mathbf{c}}_{\mathbf{k}})}{\sum_{\mathbf{c}}\sum_{\mathbf{k}}\mathbf{k}^4|\mathbf{u}^{\mathbf{c}}_{\mathbf{k}}|^2} \qquad (*: \mathbf{RNS^N})$$

where $\mathrm{t}^c_{\mathbf{k}}(\mathbf{u})$ comes from the non-linear term in the eq.(#).

The **R** *in the name* RNS *stands to stress that the equation* $\mathbf{RNS^N}$ *is time reversible, unlike the irreversible* $\mathbf{INS^N}$.

Physical interpr.: thermostats are forces with the effect of removing heat generated by the forcing.

For an incompressible fluid above, heat has to be taken away (in either enstrophy or in energy thermostat)
to maintain the relation between pressure and temperature at constant density
as prescribed by the equation of state.

The stationary distributions for the equation, referred as RNS^N, with α in (*) are parameterized by the enstrophy value D as $\boldsymbol{\lambda}_{\mathbf{D}}^{\mathbf{N}}$ and their collection will be called "enstrophy ensemble", $\mathcal{E}^N_{enstrophy}$.

Given viscosity $\boldsymbol{\nu}$ suppose, for simplicity, that there is only one SRB distribution $\boldsymbol{\mu}_{\boldsymbol{\nu}}^N \in \mathcal{E}^N_{viscosity}$ for all N large:

Conjecture: *Let $D = \boldsymbol{\mu}_{\boldsymbol{\nu}}^N(\mathcal{D})$ be the average enstrophy. Then also the distribution $\boldsymbol{\lambda}_D^N \in \mathcal{E}^N_{enstrophy}$ is unique. The distributions $\boldsymbol{\mu}_{\boldsymbol{\nu}}^N, \boldsymbol{\lambda}_D^N$ are equivalent in the sense*

$$\lim_{N\to\infty} \boldsymbol{\mu}_{\boldsymbol{\nu}}^N(O) = \lim_{N\to\infty} \boldsymbol{\lambda}_D^N(O) \qquad (@)$$

for all local observables O.

In other words the viscosity ensemble and the enstrophy ensembles are equivalent in the limit $N \to \infty$ provided their entrophies agree, if the stationary distr. is unique.

More generally the conjecture is interpreted as saying that the SRB distributions for the INS^N equation can be put in one-to-one correspondence with the distributions for the RNS^N equation with the same enstrophy so that for corresponding distributions @ holds.

First test is a non trivial consequence: namely if both sides of INS^N or RNS^N are multipled by $\overline{u}^c_{\mathbf{k}}$ and summed over $c, \mathbf{k}$ one finds, respectively:

$$\frac{d}{dt}\mathcal{E}(\mathbf{u}) = -\nu\mathcal{D}(\mathbf{u}) + \mathbf{f}\cdot\mathbf{u}, \qquad \frac{d}{dt}\mathcal{E}(\mathbf{u}) = -\alpha(\mathbf{u})D + \mathbf{f}\cdot\mathbf{u}$$

(no non-linear term ! (cancellation)).

Equivalence condition is $\langle\,\mathcal{D}\,\rangle^N_{\boldsymbol{\nu}} = D$ and $O = \mathbf{f}\cdot\mathbf{u}$ is a local observable : hence it follows that the averages $\langle\,\mathbf{f}\cdot\mathbf{u}\,\rangle$ must be equal in the limit $N\to\infty$ and

$$\boldsymbol{\nu} = \lim_{N\to\infty}\langle\,\boldsymbol{\alpha}\,\rangle^N_D$$

because the averages of $\frac{d}{dt}\mathcal{E}(\mathbf{u})$ must vanish.

Comments

(1) Equivalence test: $\langle\, \alpha \,\rangle_D^N \xrightarrow[N\to\infty]{} \nu$ is performed in 2D and 3D: with positive results in all published cases: see Fig.4 in [15] and Fig.1 in [18], Fig.4 in [19], Fig.15a in [17].

(2) The 2D tests have shown that in many cases equivalence holds also for observables that are non local. Remarkable is the observable $\alpha(\mathbf{u})$ studied as an observable for the INS^N equation. It also averages to $\boldsymbol{\nu}$ while presenting smaller fluctuations compared to the RNS^N, see [15] 2D case and 3D: Fig.16a [17] with exception in Fig.4 of [19];

(3) led to equivalence tests of other typically non local observables. A few tests, only in 2D so far, have been performed comparing, under the equivalence condition, the spectra of the symmetric part $J(\mathbf{u})$ of the $D_N \times D_N$ Jacobian matrix $\frac{\partial \dot{u}^c_{\mathbf{k}}}{\partial \dot{u}^b_{\mathbf{h}}} \stackrel{def}{=} J(\mathbf{u})_{c,\mathbf{k};b,\mathbf{h}}$.

Such observables are related to the Lyapunov exponents, [20, 21]. The result has been that essentially the eigenvalues *averaged over the flows* **agree if ordered in the same way (*e.g.* in decreasing order): see Fig.7 in [18] and Fig.5 in [15].**

Most remarkable is that, while the average of the eigenvalues agree surprisingly well, the eigenvalues of the $J(\mathbf{u})$ reach equal averages, along the two evolutions, in spite of much larger fluctuations in the RNS^N compared to the INS^N, see Fig.6 in [18].

(4) 3D tests are still somewhat preliminary: yet yield important informations. If conjecture is OK it is expected that in RNS^N the fluctuating viscosity α fluctuates and events with $\alpha < 0$ occur.

Otherwise it can be proved that D being bounded ($\nu\langle\,\mathcal{D}\,\rangle^N \xrightarrow[N\to\infty]{} = \varepsilon < \infty$, at fixed f) would imply that the velocity u remains smooth with all derivatives bounded uniformly in N, see [17, Appendix], thus giving a new prespective to the question of existence and regularity of the NS flows.

It is surprising, if ν is so small that the fluid is certainly in a turbulent regime, that for N large velocity fields $\mathbf{u}(t)$ with $\alpha(\mathbf{u}(t)) < 0$ are not observed (after a short transient time depending on the initial data) in several 3D simulations [19, 17].

Question to be understood is whether events with $\alpha < 0$ are not seen because they are rare events (which is my expectation), so rare to be missed (when N is large) in time series with too large time and/or integration step. For evidence, see Fig.15 in [17].

(5) Results in [17] suggest that conjecture above is too strong and might fail unless the definition of local observable is *deeply modified* **restricting the notion of local observable, for the purpose of the conjecture.**

So far the requirement for locality is that O **depends only on a finite number of harmonics** $\mathbf{u_k}$**: hence equivalence would be claimed for** O **depending on a single Fourier component k with** $|\mathbf{k}| > k_\nu = \ell_K^{-1}$ *if N is large enough*.

BUT..

But from [17] it emerges that equivalence is not verified in several such tests: a further condition appears needed, *i.e.* that O depends only on the components $\mathbf{u_k}$ with $|\mathbf{k}| < c_0 k_\nu$ for some constant c_0 of order 1.

See "Conjecture 2" and Fig.11–13 in [17] which suggest a value $c_0 \sim \frac{1}{8}$. The evidence is not yet conclusive, in my view: more detailed analysis is needed to exclude $c_0 = \infty$.

(6) Tests of existence of several attractors have shown that even in presence of Chaotic motions there are cases in which multiple attractors can coexist showing strong intermittency phenomena., see figure below.

(7) The remark (4) suggests that the theory of the NS equation based on searching for existence and uniqueness in function spaces could be usefully extended to equivalent equations:

while not simplifying the problem it can open perspectives, just like introducing new equilibrium ensembles does not solve basic problems of SM but, actually, introduces new ones overcompensated by the deeper understanding of thermodynamics.

Quoted references

[1] **W. George.**
Lectures in Turbulence for the 21st Century.
Lecture_Notes/Turbulence_Lille/TB_16January2013.pdf.
https://www.turbulence-online.com/Publications/, Chalmers University of Technology, Gothenburg, Sweden, 2013.

[2] **C. Fefferman.**
Existence & smoothness of the Navier–Stokes equation.
The millennium prize problems. Clay Mathematics Institute, Cambridge, MA, 2000.

[3] **D. Ruelle.**
Chaotic motions and strange attractors.
Accademia Nazionale dei Lincei, Cambridge University Press, Cambridge, 1989.

[4] **D. Ruelle.**
Dynamical systems with turbulent behavior, volume 80 of *Lecture Notes in Physics*.
Springer, 1977.

[5] **D. Ruelle.**
Turbulence, strange attractors and chaos.
World Scientific, New-York, 1995.

[6] **D. Ruelle.**
Statistical Mechanics, Rigorus Results.
World Scientific, 3d edition, London, 1999.

[7] **E. Lieb and J. Lebowitz.**
The Constitution of Matter: Existence of Thermodynamics for Systems Composed of Electrons and Nuclei.
Advances in Mathematics, 9:316–398, 1972.

[8] G. Gallavotti.
Statistical Mechanics. A short treatise.
Springer Verlag, Berlin, 2000.

[9] D. J. Evans and G. P. Morriss.
Statistical Mechanics of Nonequilibrium Fluids.
Academic Press, New-York, 1990.

[10] W. Hoover.
Time reversibility Computer simulation, and Chaos.
World Scientific, Singapore, 1999.

[11] T. Yuge, N. Ito, and A. Shimizu.
Nonequilibrium molecular dynamics simulation of electric conduction.
Journal of the Physical Society of Japan, 74:1895–1898, 2005.

[12] S. Nosé.
A unified formulation of the constant temperature molecular dynamics methods.
Journal of Chemical Physics, 81:511–519, 1984.

[13] C. Dettman and G. Morriss.
Hamiltonian formulation of the Gaussian isokinetic thermostat.
Physical Review E, 54:2495–2500, 1996.

[14] Z.S. She and E. Jackson.
Constrained Euler system for Navier-Stokes turbulence.
Physical Review Letters, 70:1255–1258, 1993.

[15] G. Gallavotti.
Nonequilibrium and Fluctuation Relation.
Journal of Statistical Physics, 180:1–55, 2020.

[16] G. Gallavotti.
Viscosity, Reversibillity, Chaotic Hypothesis, Fluctuation Theorem and Lyapunov Pairing.

Journal of Statistical Physics, 185:21:1–19, 2021.

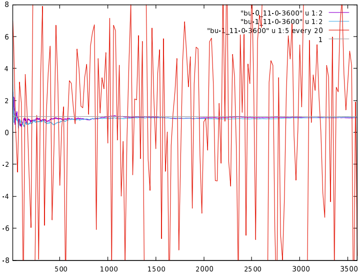
[17] G. Margazoglu, L. Biferale, M. Cencini, G. Gallavotti, and V. Lucarini.
Non-equilibrium ensembles for the 3d navier-stokes equations.
arXiv: physics.flu-dyn 2201.00530, in print on Physical Review E, 2022.

[18] G. Gallavotti.
Ensembles, Turbulence and Fluctuation Theorem.
European Physics Journal, E, 43:37, 2020.

[19] A. Jaccod and S. Chibbaro.
Constrained Reversible system for Navier-Stokes Turbulence.
Physical Review Letters, 127:194501, 2021.

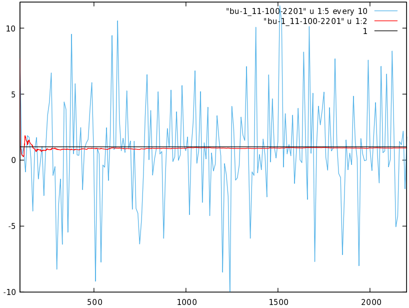
[20] D. Ruelle.
Characteristic exponents and invariant manifolds in hilbert space.
Annals of Mathematics, 115:243–290, 1982.

[21] E. Lieb.
On characteristic exponents in turbulence.
Communications in Mathematical Physics, 92:473–480, 1984.



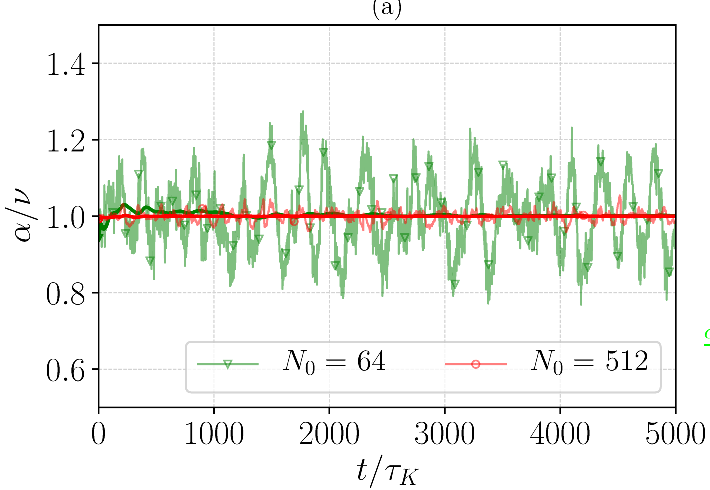
$\frac{\alpha(t)}{\nu}$, **and** $\frac{\langle\,\alpha\,\rangle_0^t}{\nu}$

Fig.4 [15] 2D: $R = 2048, N = 15$, **960 modes,** $h = 2^{-17}$
1) The large fluctations are RNS^N **values** $\frac{\alpha(t)}{\nu}$, **red,**
2) and their running average (*i.e.* time average $\frac{1}{t}\int_0^t \frac{\alpha(t')}{\nu}dt' = \frac{\langle\,\alpha\,\rangle_0^t}{\nu}$ **is red line "converging to 1.**
3) The blue line, also converging to 1, is the INS^N **running average of the observable** $\frac{\alpha(\mathbf{u}(t))}{\nu}$



$\frac{\alpha(t)}{\nu}$ $\frac{\langle\,\alpha\,\rangle_0^t}{\nu}$

Fig.1 in [18] 2D: $h = 2^{-14}$, $R = 2048$, $N = 31$, 3968 **modes: same meaning of previous**



$\frac{\alpha(t)}{\nu}$

Fig.15a in [17] 3D $\frac{\alpha(t)}{\nu}$**, and running average** $\langle \frac{\alpha}{\nu} \rangle_0^t$ **and ??** $\alpha > 0$**??;**
$N = 21$ **green,** $N = 170$ **red**

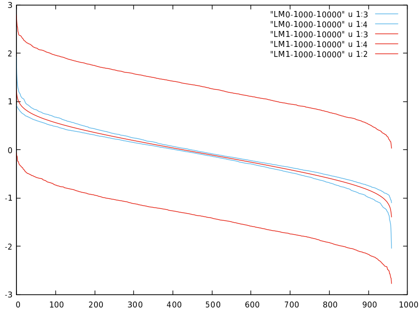


Fig.6 in [15] spectrum of Jacobian $\frac{\partial \dot{u}^c_{\mathbf{k}}}{u^r_{\mathbf{k}}}$ **in a** 960 **modes** $(N = 31)$**:** $\lambda_k(t),\ k = 0, \ldots, 959$
red$=\max_t, \min_t \lambda_k(t)$ **in RNS**
green$=\max_t, \min_t$ **in INS,**
time average in BOTH cases$=$ **central red** $\overline{\lambda}_k$**.**

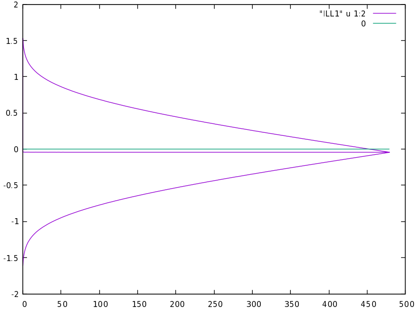
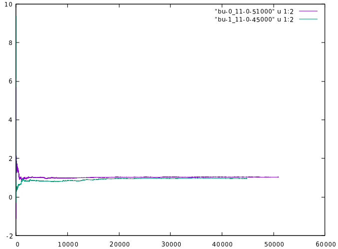


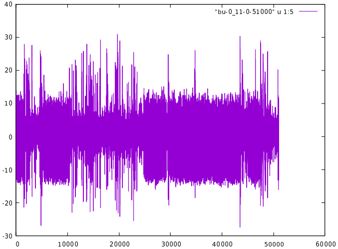
Fig.7 [18] local spectrum $\overline{\lambda}_k$

RNS & INS overlap: $R = 2048$, $h = 2^{-13}$, $N = 15,960$ **modes, the first half of the $\overline{\lambda}_k$ plotted in decreasing order while the second half is plotted in increasing order: ∼Pairing ?**



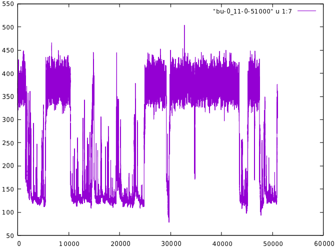
$\left\langle \frac{\alpha}{\nu} \right\rangle_0^t$

RNS & INS $R = 2048$, $h = 2^{-13}$, $N = 10$, **a case of two attractors with running average of** $\frac{\alpha(t)}{\nu}$ **nevertheless converging to 1**

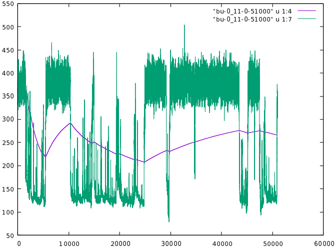


$\frac{\boldsymbol{\alpha}(t)}{\boldsymbol{\nu}}$

Fluct. of $\frac{\alpha(\mathbf{u}(t))}{\nu}$ **with 2 attractors intermittency**



$\mathbf{En(t)}$



$\langle \mathbf{En} \rangle_0^t$ $\mathbf{En(t)}$

INS $R = 2048$, $h = 2^{-13}$, $N = 10$, **previous case with intermittency of enstrophy.**