Viscosity & reversibility in fluid flows

It is well known that an exixtence-uniqueness theorem for the 3D NS-equation is an open problem.

A similar situation arises in Equilibrium Stat. Mech.

Hamiltonian equations for a system of ∞-many hard balls or Lennard-Jones atoms are (still) an open problem as far as existence-uniqueness are concerned

Yet Equilibrium Stat. Mech. (and Thermodynamics) is conceptually a problem dealing first with ∞-systems

It is flourishing not only as a Physics subject and a key for theory, experiment and applications but also as a source of deep mathematical questions and rigorous results in ODE, PDE, Probability theory, Computer science ....

But lack of existence/uniqueness of solutions for the equations of motion is not an obstacle for equilibrium SM, see [1] for a rare case in which it is solved.

The issue is simply ignored: "systems are large not infinite" and consist of N particles enclosed in **finite container** V.

Thermodynamics questions are shifted to investigating restricted properties of the finite volume stationary states which are eventually V independent.

In this context no existence-uniqueness problems arise

Mathematically the main object becomes the **state**: *i.e.* a "time-invariant" or "stationary" probability distribution $\boldsymbol{\mu}_V(d\mathbf{p}d\mathbf{q})$ on phase space.

By integration $\boldsymbol{\mu}$ gives the average values

$$\langle\, O \,\rangle_V \equiv \mu_V(O) \equiv \int O(\mathbf{p},\mathbf{q})\boldsymbol{\mu}(d\mathbf{p}d\mathbf{q})$$

of observables O which are **local functions** on phase space

i.e. depend only on the particles located in a finite region $\Lambda \subset V$ as functions of the configurations $(\mathbf{p},\mathbf{q}) = (p_1,\ldots,p_N,q_1,\ldots.q_N) \in R^{3N} \times V^{3N}$,.

The point is that at any fixed Λ and in the limit as $V \to \infty$ the averages $\boldsymbol{\mu}_V(O)$ become (or *should* become) V independent defining the **"state"**, *i.e.* the collection of averages of all local observables.

Existence of the limits as $V \to \infty$ (called "thermodynamic limit") and the properties of the **states** (*i.e.* of the collections of averages of local observables O) becomes the center of attention.

At the same time existence and uniqueness of the solutions of the underlying Hamilton equations at infinite V **does not even arise as a question**: the theory only deals with finite sistems.

A new problem, already studied by Boltzmann, is that there may be several invariant stationary states once given the macroscopic parameters like total energy E, total kinetic energy T, or temperature β^{-1} or other.

Once the governing equations are fixed, one has to deal with two immediate problems:

1) essentially all evolution ODEs and PDEs, for which unique solutions exist, admit infinitely many stationary distributions ! whereas **only one** of them, by integration, can be expected to describe the average properties of the system.

How to select it?

In Equilibrium Stat. Mech. Boltzmann and Maxwell proposed the **Ergodic Hypothesis**, EH, which typically selects the (*only*, when EH holds) distribution $\boldsymbol{\mu}$.

Out of Equilibrium, *i.e.* in presence of dissipation, the EH has been generalized to identify the "natural distribution" which is unique when the system exhibits chaos: this has been clearly formalized by Ruelle [2, 3, 4, 5] and is a simple **extension of the EH**.

2) even under the EH or its extension to chaotic systems there may remain several possible stationary distributions $\boldsymbol{\mu}$ for a given evolution eq.

Typically there are only finitely many distinct states $\boldsymbol{\mu}$ and all the others can be expressed as their convex combinations: with the interpretation of coexisting phases in St. Mech. or Intermittent phases in fluid mech. Here it is convenient to discuss omly the case in which, given the equations of motion, there is only one state.

So a real problem arises already when the natural distribution is unique but there are several equations of motion that describe the same system (as in St. Mech.): "each with its own natural distrinution"

Is this new kind on “non uniqueness” a new difficulty? is there ambiguity about observable average values ?

Solution in S.M. has been simple: such stationary distributions can be (often) shown equivalent:

They lead to the same “Thermodynamics” in the limit of infinite volume, **if** attention is limited to study properties of the mentioned class of observables: local observables.

Is it possible to follow the same path and look at the fluid equations on a similar way as the one followed in Equilibrium Stat. Mech.?

In the theory of fluids a situation arises which bears some resemblance with the above. It will be examined mainly in the case of an incompressible fluid in a periodic container.

The velocity field $\mathbf{u}(\mathbf{x})$, $\mathbf{x} \in \Omega = [0, 2\pi]^d$, $d = 2, 3$ is:

$$\mathbf{u}(\mathbf{x}) = \sum_{0<|\mathbf{k}|} \mathbf{u}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{k} = (k_\beta)_{\beta=1,\ldots d} \in Z^d$$

$$\mathbf{u}_{\mathbf{k}} = \sum_{\beta=1,2} i\, u_{\beta,\mathbf{k}}\, \mathbf{e}_\beta(\mathbf{k}), \quad \mathbf{k}\cdot\mathbf{e}_\beta(\mathbf{k}) = 0$$

with $\mathbf{e}_\beta(\mathbf{k}), \beta = 1, 2, 3$, unit vectors with $\mathbf{e}_3(\mathbf{k}) = \frac{\mathbf{k}}{|\mathbf{k}|}$ **NOT present** (*incompressibility*). And incompressible NS eq.:

$$\dot{\mathbf{u}}(x) = -(\mathbf{u}(x)\cdot\boldsymbol{\partial}_x)\mathbf{u}(x) - \nu\boldsymbol{\Delta}\mathbf{u}(x) - \boldsymbol{\partial}_x P(x),$$

$$\boldsymbol{\partial}_x \cdot \mathbf{u}(x) = 0, \quad \int_{\mathcal{T}^d} dx\, \mathbf{u}(x) = \vec{0}$$

"INS=irreversible NS", for the harmonics $u_{\beta,\mathbf{k}}$.

Here too no general existence-uniqueness (in dimension 3).

The difficulty is the lack of control of the very small scale features of the velocity fields.

In fluids interest, is directed towards properties of large classes of observables which do not depend on small scale structures, *i.e.* their value on $\mathbf{u}$ depends on the Fourier's components $\mathbf{u}_{\mathbf{k}}$ with $|\mathbf{k}| < \Lambda$ and $\Lambda \ll N$; but far from all.

Existence problem is analogous to that of existence of solutions of eq. of motion for infinite particle systems.

As in that case it will be simply set aside by imagining that the equations only regard the velocity components $u_{\beta,\mathbf{k}}$ with $|\mathbf{k}| < N$: the UV cut-off N becomes analogous to the volume cut-off V in SM.

Idea: as in SM case infinite V systems are an **abstraction**, so in FM is $N = \infty$

Of course one is **NOT interested** in the value of N, much as in SM on the size V of the volume: it should **NOT** be relevant (as long as it is large compared to the scale of the observables). This leads to propose

1) define $O(\mathbf{u})$ "local" observable if depends on $\mathbf{u}$ only through Fourier's components $u_{\beta,\mathbf{k}}$ with $|\mathbf{k}| < K$: they are supposed to evolve in time with the NS equations with UV cut-off $N \gg K$: $t \to \mathbf{u}^N(t)$

2) define a "stationary state" of the fluid the collection of the limits $N \to \infty$ of the time averaged values $\langle O \rangle_N$, along the evolution $t \to \mathbf{u}^N(t)$), for local O's:

$$\langle O \rangle = \lim_{N\to\infty} \langle O \rangle_N$$

Here $K \ll N$ is arbitrary but fixed and local observables O are alike the ones in SM depending only on particles located in a finite volume $\Lambda \ll V$.

Summarizing the parallel SM$\sim$FL:

SM : (a) local observables O depend only on particles in a finite region $K \subset V$ (arbitrary, V-independent).
(b) time evolution $\vec{x} \to \vec{x}(t)^V$ depends on vol. cut-off V.

FM : (a) local obs. O depend only on modes $\mathbf{u}_{\beta,\mathbf{k}}$ with $\mathbf{k}$ in a finite $K < N$ (arbitrary, N-independent).
(b) time evolution $\mathbf{u} \to \mathbf{u}(t)^N$ depends on UV cut-off N.

In both cases interest is on "stationary states", *i.e.* on the collection of the time averages $\langle\, O \,\rangle^V$ or, resp., $\langle\, O \,\rangle^N$ in the limits $V, N \to \infty$, (in the simple cases of no phase transitions, resp, no intermittency, *i.e.* single attractor).

To stress the analogy each collection $\mathcal{E}$ of "states" will be called an **ensemble**: as in Stat. Mech. there are *"microcanonical"*, *"canonical"*, *"grand canonical"*, *"isokinetic"*, ... ensembles

A key role is played by the stationary distributions for the regularized NS equations: their collection $\mathcal{E}^{I,N}$ will be called the **viscous ensemble** for NS.

Possibility of other ensembles and their equivalence arises if NS equations are regarded as a Hamiltonian system (Euler equations) subject to forces and to constraints ("thermostats") that impose incompressibility (*i.e.* prescribe pressure versus temperature at fixed density).

Viscosity empirically accounts for the thermostat action.

Then it should be possible to replace ν with another (as empirical) quantity achieving the same aim of allowing the evolution to generate a stationary state:

which on the selected observables will attribute the same averages in the continuum limit $N \to \infty$.

Concentrating on stationary NS fluids, the physical role of the thermostat is to avoid "blow up", due to power injected

$$\langle\, W \,\rangle = \langle\, \int_\Omega \mathbf{f}(\xi)\cdot\mathbf{u}(x)dx \,\rangle_N$$

thus allowing to reach a stationary state.

Important to keep in mind that, if $f_{\mathbf{k}}$ is a large scale forcing ($i.e.$ $|f_{\mathbf{k}}|\neq 0$ only for small $|\mathbf{k}|$) $\Rightarrow W$ is a local observable.

So we can classify stationary states by the time average of "dissipation" $\langle\, W \,\rangle$ which in the standard NS equation is proportional to the average **"enstrophy"** $\mathcal{D}(\mathbf{u})$

$$D = \langle\, \mathcal{D} \,\rangle = \langle\, \int_\Omega (\partial\mathbf{u}(x))^2 dx \,\rangle : \quad \nu D = \langle\, W \,\rangle$$

[i.e. multiply the NS equation bu $\mathbf{u}$ and integrate, finding $\frac{d}{dt}\int\frac{\mathbf{u}(x)^2}{2}dx = -\nu\int(\boldsymbol{\partial}\mathbf{u})^2 + \int\mathbf{f}\cdot\mathbf{u}dx$ and use that l.h.s has 0 average, being a time derivative, $i.e.$ $\nu D = \langle\, W \,\rangle$

A proposal for an alternative thermostat is to replace $\nu\Delta\mathbf{u}$ by $\alpha(\mathbf{u})\Delta\mathbf{u}$ with $\alpha(\mathbf{u})$ a multiplier such that the enstrophy D itself is a constant of motion. The equation, here "RNS=reversible NS", with UV cut-off at $|\mathbf{k}| \le N$:

$$\dot{\mathbf{u}} = -(\underset{\sim}{\mathbf{u}} \cdot \underset{\sim}{\boldsymbol{\partial}})\mathbf{u} - \alpha(\mathbf{u})\Delta\mathbf{u} + \mathbf{f} - \boldsymbol{\partial}P$$

And if $\Lambda(\mathbf{u}) = -\int_\Omega (\underset{\sim}{\mathbf{u}} \cdot \underset{\sim}{\boldsymbol{\partial}}\mathbf{u}) \cdot \Delta\mathbf{u}\, dx$, [multiply both sides by $\Delta\mathbf{u}$ integrate by parts find α to obtain 0], α turns out:

$$\alpha(\mathbf{u}) = \frac{\Lambda(\mathbf{u}) + \sum_{\mathbf{k}} \mathbf{k}^2 \mathbf{f}_{\mathbf{k}} \cdot \overline{\mathbf{u}}_{\mathbf{k}}}{\sum_{\mathbf{k}} \mathbf{k}^4 |\mathbf{u}_{\mathbf{k}}|^2}, \ d = 3$$

Given a $\mathbf{f}$ "large scale forcing" , $\mathbf{f}_{\mathbf{k}} \ne 0, |\mathbf{k}| < k_0, \|\mathbf{f}\| = 1$ the equation in $\Omega = [0, 2\pi]^d$ has only one parameter: the viscosity ν for INS and the enstrophy D for RNS.

Fixed the UV cut-off N, as ν or D vary let

$\mu_\nu^{I,N}(d\mathbf{u})$= stationary distribution for INS, parameter ν

$\mu_D^{R,N}(d\mathbf{u})$= stationary distribution for RNS parameter D

and introduce two ensembles:

"viscosity ensemble" be collection $\mathcal{E}^{I,N}$ of the $\mu_\nu^{I,N}(d\mathbf{u})$

"enstrophy ensemble" be collection $\mathcal{E}^{R,N}$ of the $\mu_D^{R,N}(d\mathbf{u})$

The proposal is: there should be a 1-1 correspondence between the elements of the two ensembles, (*i.e.* between ν and D) and:

in corresponding elements the expectation values of **all local** observables will coincide in the limit $N \to \infty$.

Formally:

Equivalence conjecture ("EC"): *Under equal dissipation condition, i.e. if* ν *and* $\mathcal{D}(\mathbf{u})$ *verify*

$$\langle\, \mathcal{D} \,\rangle_\nu^{i,N} = D$$

it is, for all local observables O*:*

$$\langle\, O \,\rangle_\nu^{I} \stackrel{def}{=} \lim_{N\to\infty} \langle\, O \,\rangle_\nu^{I,N} = \lim_{N\to\infty} \langle\, O \,\rangle_D^{R,N} \stackrel{def}{=} \langle\, O \,\rangle_D^{R}$$

for all local observables O. [6, 7, 8]

In particular $W = \sum_{\mathbf{k}} f_{\mathbf{k}} \overline{u}_{\mathbf{k}}$ is a local observable and therefore $\langle\, W \,\rangle_\nu^{i,N} = \langle\, W \,\rangle_D^{r,N}$: at corresponding ν, D (being averages of the local observable $W(\mathbf{u})$); but

$$\langle\, W \,\rangle_D^{I,N} = \nu \langle\, \mathcal{D} \,\rangle, \quad \langle\, W \,\rangle_D^{R,N} = \langle\, \alpha \,\rangle D$$

follows from the equations: therefore:

$$\nu = \langle\, \alpha \,\rangle_D^{R,N}$$

The mentioned simple rigorous consequence of EC:

$$\nu = \lim_{N\to\infty} \mu_{En}^{r,N}(\alpha)$$

this can be a first simple, but demanding, test of the conjecture, *"viscosity test"*.

The α will show very strong fluctuations over the time scale of the largest Lyapunov exponent, at least if the viscosity is so small that a global existence cannot be ascertained for the INS without cut-off.

Still the EC implies that the time average of α is ν.

Remark:

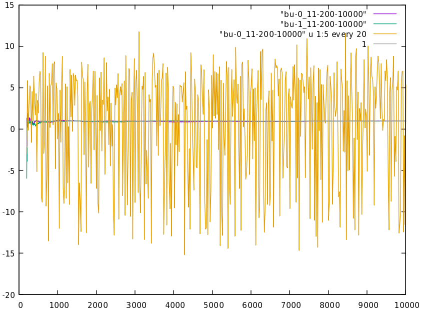
Negative values if α "must" show up if ν is small enough **at least for all N large**: otherwise the existence and uniqueness for NS would be implicitly solved, because (it is a simple theorem that) if

$\alpha(\mathbf{u}^N(t)) \geq \varepsilon > 0$ *for all N large (eventually in t) and for some ε the UV cut-off equation would have N-uniformly* **smooth** *solutions with probability 1 in stationary states*

Therefore I expect that possibly very rarely negative values of α should be observed.

Tests, only for $d = 2$ and work is in progress for $d = 3$

1) **viscosity test** (2D):



Green: $(t, \frac{1}{t}\int_0^t \frac{1}{\nu}\alpha(\mathbf{u}(\tau))d\tau)$, in RNS (the test)

Red: $(t, \frac{1}{t}\int_0^t \frac{1}{\nu}\alpha(\mathbf{u}(\tau))d\tau)$, in INS, (expected?)

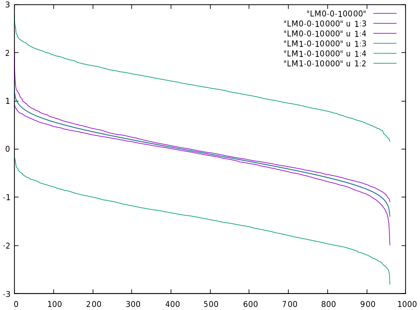
Yellow: value of $(t, \frac{1}{\nu}\alpha(\mathbf{u}(t)))$ in INS, ($\sim$same in RNS !)

Black: conjectured value 1

integration step $h = 2^{-17}$, recorded every $4h^{-1}$ steps

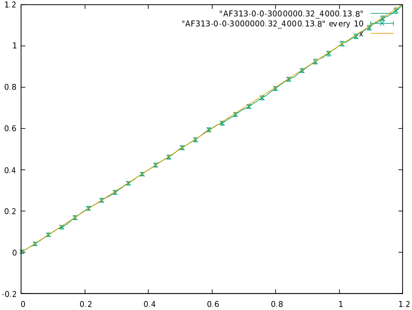
time unit 4, total time 10^4 records, $N = 31 \times 31$

2) Lyapunov test: (2D) (not part of conjecture)



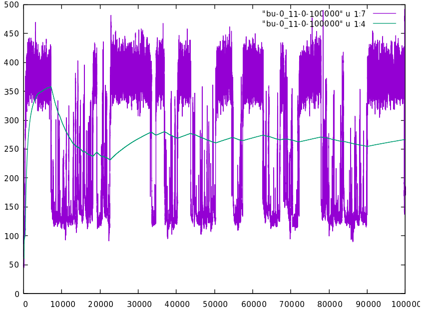
Red: Local Lyapunov exp. $(k, \max_t \lambda_k(t))$ and $(k, \min_t \lambda_k(t))$ for INS
Green: Local Lyapunov exp. $(k, \max_t \lambda_k(t))$ and $(k, \min_t \lambda_k(t))$ for RNS
Blue: common average value of INS and RNS exponents
31×31 resolution.
expected?? NO!

3) **Reversibility test on INS** (expected ??)



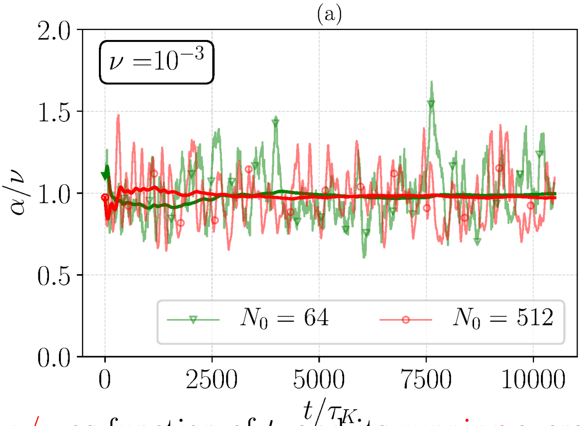
Chaotic hypothesis implies "Fluctuation theorem": if $\sigma(\mathbf{u})$=volume contraction rate with time average σ_+, the $p = \frac{1}{t}\int_0^t \frac{\sigma(\mathbf{u}(\tau))}{\sigma_+} d\tau$ (average "entropy production rate") satisfies large deviation law at rate $s(p)$ s.t. $(s(p) - s(-p)) = p$. Graph is over $3 \cdot 10^6$ INS evolution data **BUT** uses $\sigma(\mathbf{u})$ the reversible observable: *i.e.* irreversible flow looks reversible. 7×7 resolution.

4) **Intermittency test** (2D)



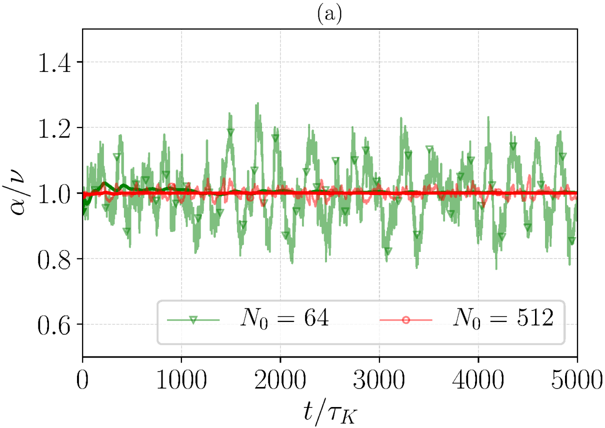
enstrophy as function of t, and its running average, in INS shows intermittency. BUT a check shows that the F.T. does not hold in this case.
There are rather long intervals during which the motion seems to dwell on a single attractor and FT might hold.
15×15-resolution. Problem of precision?

5) Preliminary: 3D viscosity test, RNS[9]



α/ν as function of t, and its running average, In solid line, green (for UV cut off $N = 21$, *i.e.* 42^3 Fourier's modes) and red (for $N = 166$, *i.e.* 336^3 modes) is the running average of α/ν. Here $N \simeq N0/3$.

6) Preliminary 3D viscosity test INS, ecpected?[9]



It is important to remark that in the above cases the values of α stay substantially > 0. In [9] it was not possible to exclude convincingly whether values $\alpha < 0$ were absent with probability 1 in the stationary states $\boldsymbol{\mu}$ or not.

This might be because $\alpha < 0$ has, in the natural states $\boldsymbol{\mu}$ an extremely small probability (as I believe).

Also in [9] the Equivalence conjecture has been strongly weakened by limiting the class of the local observables which would have equal averages in the two ensembles $\mathcal{E}^{I,N}$ and $\mathcal{E}^{R,N}$.

Namely only O localized with $|\mathbf{k}| < \Lambda$ can have equal averages, in corresponding states in the two ensembles, if $\Lambda < ck_\nu$ where K_ν is the Kolmogorov's scale $k_\nu = (\frac{\langle W \rangle}{\nu^3})^{\frac{1}{4}}$ and $c_\nu \xrightarrow[\nu\to 0]{} c_0 \le \infty$, ($\langle W \rangle = \nu D$= dissipation per unit time).

In [9] the limitation has been forced by the computer non-availability to probe convincingly (in a reasonable time and accuracy) the conjecture at larger cut-off values: the possibility that the conjecture holds, in the form stated here, remains while waiting for more powerful computers.

(1): The idea of "equivalent thermostats" goes back, in SM, to [10, 11], and received important contributions from the Sydney school, [12, 13].

(2): Application to fluids, in a somewhat different form and context, appeared in [14],

(3): In the "weak" form, fixed N and $\nu \to 0$, was discussed in [15, 16, 17, 18, 19, 7, 20] and

(4): In the form discussed here, particularly relevant for fluids, in [7, 8, 21, 9].

(5): One can also consider other ensembles: for instance defining the multiplier $\alpha(\mathbf{u})$ so that energy rather than enstrophy is conserved. The latter ensemble has been considered, in view of the equivalence, in [22] for 3DNS, with remarkable results.

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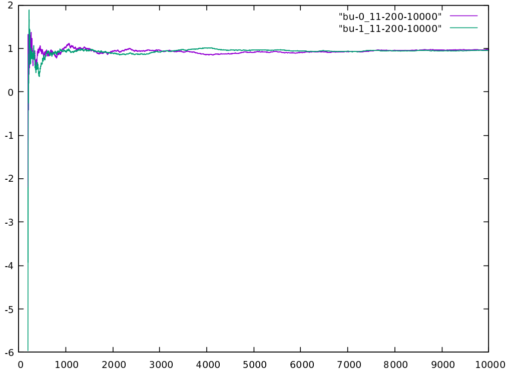
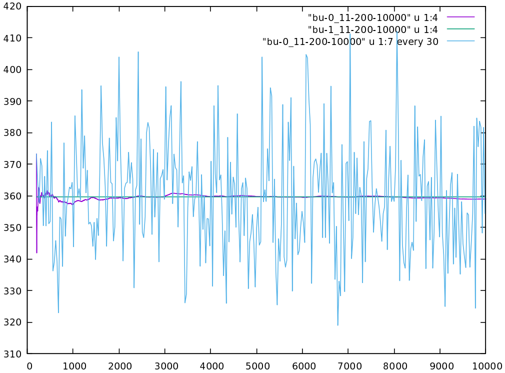
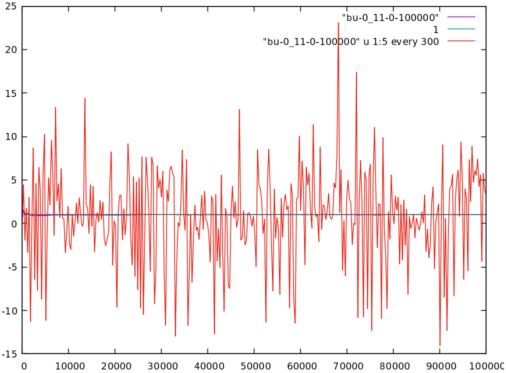
Supplementary material



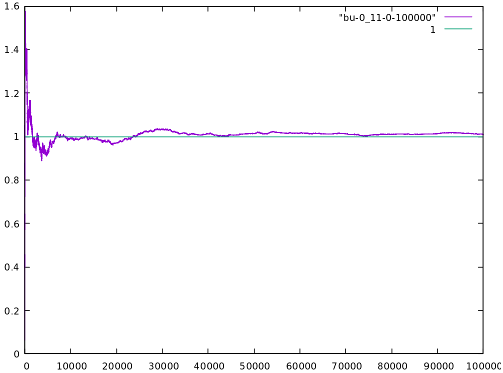
Same viscosity test: only running average of $\frac{\alpha}{\nu}$, 31×31



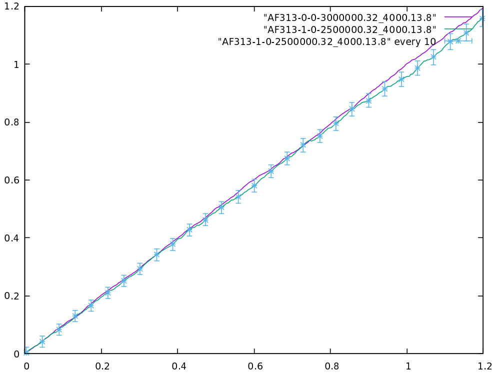
Enstrophy running average and fluctuations in INS and superposed the enstrophy of the equivalent RNS; 31×31



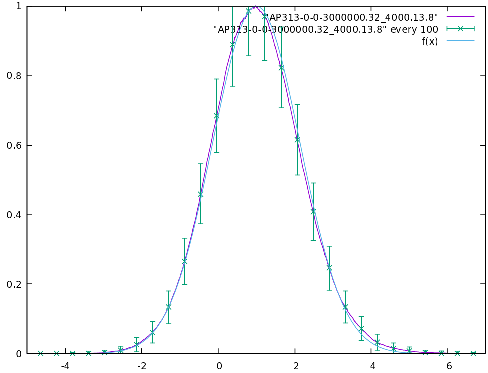
Viscosity test in intermittent flow; 15×15



Same with only running average of $\frac{\alpha}{\nu}$



FT: p-fluctuation for INS and, with error bars, RNS.



FT: p-distribution for INS and, with error bars, RNS. Compatible with Gaussian.