Chaotic Hypothesis (CH):
A chaotic flow (M, S_t) has attracting&transitive&hyperbolic surfaces $\mathcal{A}$, [1, 2]

Navier-Stokes, friction ν, incompressible & periodic b.c.:

$$\dot{\mathrm{u}}(x) = -(\underset{\sim}{\mathrm{u}} \cdot \underset{\sim}{\partial})\mathrm{u} + \nu\Delta\mathrm{u} + \mathrm{f} - \partial p, \qquad \boldsymbol{\partial} \cdot \mathrm{u} = 0 \qquad NS$$

f fixed and long range.
UV-N Regularization:

$$\mathrm{u}(x) = \frac{1}{(2\pi)^3} \sum_{|\mathrm{k}| \le N, \gamma = \pm 1} u_{\mathrm{k},\gamma} \mathrm{e}_{\mathrm{k},\gamma} e^{i\mathrm{k}\cdot x}$$

$$\mathrm{e}_{\mathrm{k},\pm} = -\,\mathrm{e}_{-\mathrm{k},\pm} \quad \text{real elicities}, \qquad u_{\mathrm{k},\gamma} = \overline{u}_{-\mathrm{k},\gamma}$$

Stationary non equilibrium[3]
Unique statistical dist. $\mu^N(d\mathrm{u})$ on $\mathcal{A}$'s, 'SRB'

Viscosity ν is statistical: dissipation $\nu\mathcal{D} = \nu\sum_{|\mathsf{k}|\le N,\gamma}\mathsf{k}^2|\mathsf{u}_{\mathsf{k},\gamma}|^2$; balances $\mathsf{f}\cdot\mathsf{u}$: it can be imagined fluctuating, ν being replaced by

$$\nu \quad\Rightarrow\quad \begin{array}{ll} \alpha(\mathsf{u}) = \dfrac{(\Delta\mathsf{u}\cdot\mathsf{f})}{(\Delta\mathsf{u})^2}, & 2\ \mathrm{dim.}\ \mathit{RNS} \\ \alpha(\mathsf{u}) = \dfrac{[(\underset{\sim}{u}\cdot\underset{\sim}{\partial})\mathsf{u}]\cdot\Delta\mathsf{u} + (\Delta\mathsf{u}\cdot\mathsf{f})}{(\Delta\mathsf{u})^2}, & 3\ \mathrm{dim.}\ \mathit{RNS} \end{array}$$

The new eq. conserves enstrophy $\mathcal{D}$ and is **time-reversible**.

Call ***En*** **average of $\mathcal{D}$** in the stationary state of NS and $\overline{\alpha}$ **average of α** in the reversible equation RNS

Correspondent states: are stationary distributions μ_ν^N distrib. for the NS eq. and $\widetilde{\mu}_{En}^N$ for the RNS if:

$$\overline{\alpha} \stackrel{def}{=} \widetilde{\mu}_{En}^N(\alpha) = \nu \qquad \mathit{or} \quad \mu_\nu^N(\mathcal{D}) = \mathit{En}$$

Correspondent means *equal average dissipation*.

Let $F(\mathrm{u})$ depend on finitely many $\mathrm{u}_{\mathrm{k},\gamma}$: the observable will be called **local** or **large scale**.

The forcing will be fixed, once and for all, also to be local

Conjecture: If given En and ν the states $\widetilde{\mu}^N_{En}$ and μ^N_ν are correspondent then for all local observables

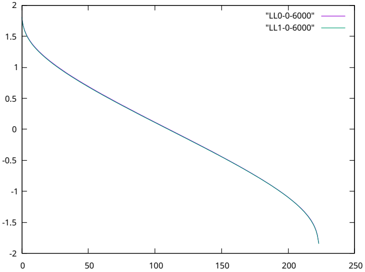
$$\lim_{N\to\infty} \mu^N_\nu(F) = \lim_{N\to\infty} \widetilde{\mu}^N_{En}(F)$$

Remarks: *Similarity with the thermod. limit canonical-grand canonical equivalence. As in Stat.Mech. there are other ensembles; here similar considerations apply if α is designed to keep for instance the total energy constant*

Tests of the conjecture done in **2D**, *[4], and weaker forms have been considered to accomodate* **3D** *simulations, [5], (which otherwise could be interpretable as negative results). The matter is at the moment open.*

Question: equivalence for nonlocal observ.? (as in Stat. Mech.)

Interesting non local observables are the Lyapunov exp. and the first of a series and it can be checked in a short time. In 2-dim. experiments : comparison between the Lyapunov spectra



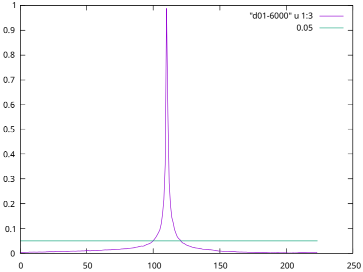


Fig.1: Lyap.-spectra NS-RNS: This particular case: 224 d.o.f, viscosity 2^{-11}, int. step (time step $h = 2^{-17}$), forcing acting on a single $\pm$k. It gives spectra close within 4% (away of $\lambda_i \simeq 0$, of course). Checks equivalence of spectra NS-RNS: $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{\mathcal{N}-1}$, in dDim $\mathcal{N} = 2^{d-2}((2N+1)^d - 1)$. Many new features emerge.

A remarkable **pairing symmetry** appears in tests with N small, [6]

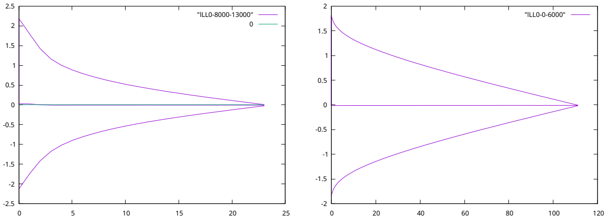


Fig.2: Lyap.-spectra NS-RNS: **Pairing** of Lyap. exp.: 48 and 224 d.o.f: $\lambda_j + \lambda_{\mathcal{N}-1-j} = const$ and const $\simeq 0$. Q.: Contradicts viscous dissip. ?

Unexpexted **(?)** in RNS and NS. In the RNS it is consistent with Kaplan-Yorke dimension = full and therefore the chaotic hypothesis should imply the Fluctuation Th.: it induced test FT in the above 48 d.o.f, [7]: one of the very few tests of FT.

The const=0 is **BUT suspect**: should depend on UV cut-off N. Clearly if N is increased negative friction on small scales will prevail and dimension of attracting surface will **go down** and with it **flow reversibility, Fluct. Th., pairing to** 0**, ...**

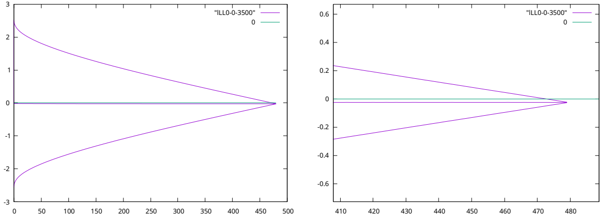


Fig.3: Pairing of Lyap. exp.: 960 d.o.f ($N = 15$). The pairing remains **but at level <0**; KY dimension $d = 900 < 960$, [8].

Tempting to interpret that attracting surface $\mathcal{A}$ is 900-dim. However **why pairing at** < 0 **value**? The explanation I propose is that the flow **on** $\mathcal{A}$ **remains reversible** (with pairing at 0), but increasing further N the **complete Lyapunov spectrum** will give pairing, not to a constant but to a curve concave like a parabola.

Reversibility on the attracting surface has been proposed and called "Axiom C", [9]. Would strongly support equivalence NS-RNS.

However it **would not be** time reversal symmetry (**TRS**) of the equations of motion **!** which tranforms the attracting surface into a (disjoint) **repelling one**. The TRS would be spontaneously broken and replaced by a new symmetry whose existence is provided by the axiom C, if accepted, [10].

Of course the future will need **test in simulations** of all the properties implied by the CH, axiom C: an example beyond the cases mentioned above it that the FT might hold but with a different slope (in the 960 d.o.f. case above would be $\frac{900}{960}$).

The fluctuations are around average Lyap. exp. in NS and RNS.

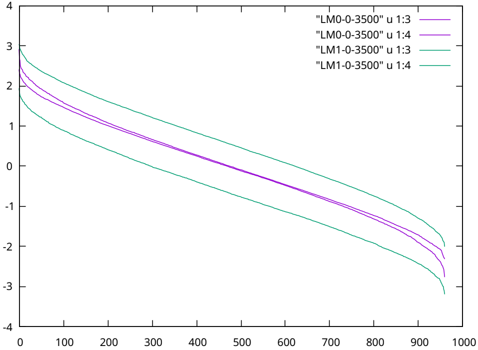


Figure: The two inner lines give of each $k = 0, \ldots, 959$ the **maximum and minimum** of λ_k computed in 3500 iterations in the NS flow. The two external lines give the same for the RNS flow: the averages agree as in the previous drawing, but markedly different fluctuations

Quoted references

[1] G. Gallavotti and D. Cohen.
Dynamical ensembles in nonequilibrium statistical mechanics.
Physical Review Letters, 74:2694–2697, 1995.

[2] G. Gallavotti and D. Cohen.
Dynamical ensembles in stationary states.
Journal of Statistical Physics, 80:931–970, 1995.

[3] D. Ruelle.
Smooth dynamics and new theoretical ideas in non-equilibrium statistical mechanics.
Journal of Statistical Physics, 95:393–468, 1999.

[4] G. Gallavotti.
Finite thermostats in classical and quantum nonequilibrium.
European Physics Journal Special Topics, 227:217–229, 2018.

[5] G. Margazoglu, L. Biferale, M. Cencini, G. Gallavotti, and V. Lucarini.
Non-equilibrium Ensembles for the 3D Navier-Stokes Equations.
Physical Review E, 105:065110, 2022.

[6] G. Gallavotti, L. Rondoni, and E. Segre.
Lyapunov spectra and nonequilibrium ensembles equivalence in 2d fluid.
Physica D, 187:358–369, 2004.

[7] G. Gallavotti.
Ensembles, Turbulence and Fluctuation Theorem.
European Physics Journal, E, 43:37, 2020.

[8] P. Constantin and C. Foias.
Global Lyapunov Exponents, Kaplan-Yorke Formulas and the Dimension of the Attractors for 2D Navier-Stokes Equations.
Retrieved from the University of Minnesota Digital Conservancy, http://hdl.handle.net/11299/4254:1–47, 1983.

[9] F. Bonetto and G. Gallavotti.
Reversibility, coarse graining and the chaoticity principle.
Communications in Mathematical Physics, 189:263–276, 1997.

[10] G. Gallavotti.
A local fluctuation theorem.
Physica A, 263:39–50, 1999.