Fermions & Quasi-particles & RG

Let $\psi^{\pm}_{x,t} = e^{tT}\psi^{\pm}_{x}e^{-tT}$ be Fermionic fields. The infrared problem: evaluate $\mathrm{Tr}\int e^{V_I(\psi)}dP_g$ with $I = [-\frac{1}{2}L, \frac{1}{2}L]^{d+1}$

$$V_I(\psi) = \int_I \lambda_0(x-y) : \psi^+_{t,x}\psi^+_{t,y}\psi^-_{t,y}\psi^-_{t,x} : dtdxdy$$
$$+ \int_I (\alpha_0 : \psi^+_{t,x}(-\Delta_x + p_F^2)\psi^-_{t,x} : + \nu_0 : \psi^+_{t,x}\psi^-_{t,x} :)dtdx$$

by power expansion of e^{V_I} and computing $\int dP_g$ of the resulting polynomials in the fields by Wicks' rule with "propagator function" g in $d+1$ dimensions, ($\beta = \frac{p_F}{m}$).

$$\mathbf{g}(\mathbf{x}, \mathbf{t}) = \int \frac{dk_0 d\mathbf{k}}{(2\pi)^{d+1}} \frac{e^{-i(k_0 t + \mathbf{k}\mathbf{x})}}{-ik_0 + (\mathbf{k}^2 - p_F^2)/2m} \vartheta(\mathbf{k}^2/\mathbf{p}_0^2) =$$

Idea: multiscale approach $g(\mathbf{x}, t) = \sum_{n=0}^{-\infty} g^{(n)}(\mathbf{x}, t)$ with $g^{(n)}$ smooth on scale $\gamma^n \equiv 2^n, n \le 0$.

Difficulty: g depends on 2-length scales (p_0, p_F)

Introduce quasi particles fields $\psi^{\pm}_{t,\mathbf{x},\omega}$ in terms of which the *particle* **fields $\psi^{\pm}_{t,\mathbf{x}}$ are written, [1]:**

$$\psi^{\pm}_{t,\mathbf{x}} = \sum_{0}^{-\infty} \psi^{\pm,(n)}_{t,\mathbf{x}}, \qquad \psi^{\pm(n)}_{x} = \int_{|\boldsymbol{\omega}|=1} d\boldsymbol{\omega} e^{\pm i p_F \boldsymbol{\omega}\mathbf{x}} \psi^{\pm(n)}_{x,\boldsymbol{\omega}}$$

$x = (t, \mathbf{x})$ **with integral over Fermi sphere momenta $p_F\omega$.**

Quasi particles propagator $g^{(n)}(x, \omega; x', \omega')$ is chosen so that:

$$\mathbf{g^{(n)}(x, \omega; x', \omega') = \delta(\omega - \omega') g^{(n)}(x - x', \omega)}$$

$$g^{(n)}(x) = \int d\boldsymbol{\omega} e^{i p_F \boldsymbol{\omega}\mathbf{x}} g^{(n)}(x, \boldsymbol{\omega})$$

$$\mathbf{g^{(n)}(x, \omega) \sim 2^n C_d p_F^{d-1} p_0 (\tau_n - i\omega \cdot x_n) \gamma(x_n)}$$

with $\gamma(x)$ exp decreasing in $x_n = 2^n(\tau_n, \mathbf{x}_n)$ and $\frac{p_F}{m} = 1$, [1]

Note that the field dimension is 2^n in all dimensions d.

UV integration leads to a V_l of the form (***) plus a remainder sum of polynomials with coefficients of h.o. on the constants $\mathcal{L}_0 = (\lambda_0, \nu_0, \alpha_0, \zeta_0)$

$$\int \prod_{j=1}^{4} dx_j d\omega_j e^{ip_F(\sum_{i=1}^{2} \omega_i \mathbf{x}_i - \sum_{i=3}^{4} \omega_i \mathbf{x}_i)}$$

$$\lambda_{\mathbf{0}}(\boldsymbol{\omega}_{\mathbf{1}}, \boldsymbol{\omega}_{\mathbf{2}}, \boldsymbol{\omega}_{\mathbf{3}}, \boldsymbol{\omega}_{\mathbf{4}}) \psi^+_{x_1,\omega_1} \psi^+_{x_1,\omega_2} \psi^-_{x_1,\omega_3} \psi^-_{x_1,\omega_4} + (***)$$

$$+ \int \prod_{j=1}^{2} dx_j d\omega_j e^{ip_F(\omega_1 \mathbf{x}_1 - \omega_2 \mathbf{x}_2)}$$

$$\psi^+_{x_1,\omega_1} [\boldsymbol{\nu}_0 + \boldsymbol{\alpha}_0 \mathcal{D}_{\mathbf{x},\boldsymbol{\omega}_2} + \boldsymbol{\zeta}_{\mathbf{0}} \partial_{\mathbf{t}}] \psi^-_{x,\omega_2} + UV$$

where $\mathcal{D}_{\mathbf{x},\omega} \equiv \partial_{\mathbf{x}} - i \frac{\omega}{2p_F} \Delta_{\mathbf{x}}$.

Perform the expansion

$$\int \prod_{n=0}^{-N} dP_g^{(n)} (1 + V_0 + \frac{1}{2} V_0^2 + \frac{1}{3!} V_0^3 + \dots)$$

Applying Wick's th. express integrals at scale $n=0$ as sum of Wick ordered monomials with coefficients polynomials in $\mathcal{L}_0=(\lambda_0,\alpha_0,\nu_0)$. Introduce "localization operators"

$$\mathcal{L}:\psi^+_{\mathbf{x_1},\omega_1}\psi^-_{\mathbf{x_2},\omega_2}:=:\psi^+_{\mathbf{x_1},\omega_1}(\psi^-_{\mathbf{x_1},\omega_2}+(\mathbf{x_2}-\mathbf{x_1})\widetilde{\mathcal{D}}_{\omega_2}\psi^-_{\mathbf{x_1},\omega_2}):$$

$$\mathcal{L}:\psi^+_{\mathbf{x_1},\omega_1}\psi^+_{\mathbf{x_2},\omega_2}\psi^-_{\mathbf{x_3},\omega_3}\psi^-_{\mathbf{x_4},\omega_4}:=\frac{1}{2}\sum_{\mathbf{j}=1,2}:\psi^+_{\mathbf{x_j},\omega_1}\psi^+_{\mathbf{x_j},\omega_2}\psi^-_{\mathbf{x_j},\omega_3}\psi^-_{\mathbf{x_j},\omega_4}:$$

where $\widetilde{\mathcal{D}}_{\omega}=(\partial_t,\mathcal{D}\omega)$. Resulting Wick's monomials of order $1,2,4$ in $\psi^{\pm}_{x,\omega}$ will have the form below with coefficents $\Lambda_{-1}=(\lambda_{-1},\alpha_{-1},\nu_{-1})$. Iterating power series result:[2, 3]; *e.g.* **if $\mathbf{d}=\mathbf{1}$:**

$$\begin{aligned}
\lambda_{h-1}=&\lambda_h+\lambda_h^3G_1(\lambda_{\ge h})+\alpha_h\lambda_h^3G_2(\lambda_{\ge h},\alpha_{\ge h})+\nu_h^2\lambda_h^2G_3(\lambda_{\ge h},\alpha_{\ge h},\nu_{\ge h})+\\
&+2^hR_1(\lambda_{\ge h},\alpha_{\ge h},\nu_{\ge h})\\
\alpha_{h-1}=&\alpha_h+\lambda_h^2\alpha_hG_4(\lambda_{\ge h},\alpha_{\ge h})+\nu_h^2G_5(\lambda_{\ge h},\alpha_{\ge h},\nu_{\ge h})+2^hR_2(\lambda_{\ge h},\alpha_{\ge h}\\
\nu_{h-1}=&2\nu_h+\nu_h\lambda_h^2G_6(\lambda_{\ge h},\alpha_{\ge h},\nu_{\ge h})+2^hR_3(\lambda_{\ge h},\alpha_{\ge h},\nu_{\ge h})
\end{aligned}$$

At fixed $h \leq 0$ the series in the couplings Λ_h exist if $|\Lambda_h| < C$; and satisfy, at order n, scale indep. bounds $DC^n n!$.

Two problems. Deepest is that there is $n!$ in the estimates of the n-th order coefficients of B functions. And even heuristically the analysis of the dyn. system generated by the map $\Lambda_h \to \Lambda_{h-1}$ is difficult, [1].

Better results: for spinless fermions at $d = 1$ (hence ω takes only two values, ± 1), power series for the R_j, G_j actually converge; but the second problem remains.

Key idea at $d = 1$: use exact solubility of Luttinger's model.

Studying the model with the same formalism as above, it was possible to realize the identity of the G_1 beta function part, [3], in the two models.

And G_1 thad to be exactly 0 to be compatible with the exact solution.

This was made possible by the hard work of Benfatto on the precise study of the beta function, [1]: if λ_0 is small enough, α_0, ν_0, ζ_0 can be assigned so Λ_h converges to a limit as $h \to -\infty$.

It also led to the proof of anomalous scaling in 1D models, as conjectured in [3].

At the time remained questions of mathematical rigor which, again with the essential contribution of Benfatto, were settled in shortly subsequent works, [4].

Other work in which I had the privilege of collaborating with Benfatto and his wealth of new ideas are:

UV stability of Euclidean φ^4, $d = 2, 3$, [5, 6].

Smoothness properties of Gaussian fields in dimensions 2, [7].

Disorder in the 1D spinless Holstein model [?]

Kondo effect in a fermionic hierarchical model [8]

Some of the above results have been summarized in the book Renormalization group [9].

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His personality and width of interests is made more evident by his works with other colleagues and witnesses his strong inflence on the Mathematical-Physics groups at Roma.

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$$
\begin{aligned}
\lambda_{h-1} =& \lambda_h + B_1^{(\geq 0;[1,2])}(\alpha_h, \zeta_h, \nu_h; \lambda_h) + 2^{\varepsilon h} B_2^{(\geq 1;\geq 1)}(\alpha_h, \zeta_h, \nu_h; \lambda_h), (\varepsilon <\\
\nu_{h-1} =& 2\nu_h + 2^{\varepsilon h} B_3^{(\geq 2)}(\alpha_h, \zeta_h, \nu_h, \lambda_h)\\
\alpha_{h-1} =& \alpha_h + \beta'' \nu_h^2 + 2^{\varepsilon h} B_4^{(\geq 2)}(\alpha_h, \zeta_h, \nu_h, \lambda_h)\\
\zeta_{h-1} =& \zeta_h - \beta'' \nu_h^2 + 2^{\varepsilon h} B_5^{(\geq 2)}(\alpha_h, \zeta_h, \nu_h, \lambda_h)
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{h-1} =& \lambda_h + \lambda_h^3 G_1(\lambda_h) + \delta_h \lambda_h^2 G_2(\lambda_h, \delta_h) + \nu_h^2 \lambda_h^2 G_3(\lambda_h, \delta_h, \nu_h) +\\
& + t_h R_1(\lambda_h, \delta_h, \nu_h, t_h)\\
\delta_{h-1} =& \delta_h + \lambda_h^2 \delta_h G_4(\lambda_h, \delta_h) + \lambda_h^2 \nu_h G_5(\lambda_h, \delta_h, \nu_h) + t_h R_2(\lambda_h, \delta_h, \nu_h, t_h)\\
\nu_{h-1} =& 2\nu_h + \nu_h \lambda_h^2 G_6(\lambda_h, \delta_h, \nu_h, t_h) + \delta_h \lambda_h^2 G_7(\lambda_h, \delta_h, \nu_h, t_h) +\\
& + R_3(\lambda_h, \delta_h, \nu_h, t_h)\\
t_{h-1} =& 2^{-1} t_h
\end{aligned}
$$