Let $\alpha \in \mathcal{D}$ be C^∞ rapidly decreasing:

Proposition: *Let Λ be a cube and $\Lambda_1, \Lambda_2, \ldots$ be unit cubes paving Λ. Let $\alpha_1 \in C^\infty(R^d)$, with support in Λ_1 and let $\alpha_j(x) = \alpha_1(x - \xi_j)$ where ξ_j is the center of Λ_j. Then if $s < m - \frac{d}{2}$, $\exists\, c_j, j = 1.\ldots, 4$, Λ-independent constants, such that $\forall j$:*

$$\mu_A(\varphi : \big| \, \|\alpha_j \varphi\|_{H_s(R^d)} \le B) \ge e^{-c_3 e^{-c_4 B^2} |\Lambda|}$$

and if s=integer, $s + \varepsilon < m - \frac{d}{2}$ the probability that the Hölder C_ε norm $\|\alpha_j \varphi\|_{C_\varepsilon} < B$ is $\ge e^{-c_1 e^{-c_2} B^2 |\Lambda|}$

Replace R^d by a cube $\Lambda \subset Z^d$ and let μ_A be the Gaussian process with index space Λ and generator

$$A = (1 - D)^{-m}, \qquad m \in Z_+$$

where D is the discretized Laplacian.

Consider D with periodic boundary conditions on the boundary of
Λ. **Let L be the side size of Λ (so that $|\Lambda| = L^d$).**

Of course the main point is the uniformity in the size of Λ.

A basic property of Dirichlet problem: let $\Omega \subset Z^d$ and define $\partial^m\Omega = \{\xi \,|\, \xi \not\in \Omega,\, d(\sigma, \Omega) \le m\}$: *Let $\Omega \subset Z^d$ and the equation $(1-D)^m\, u = 0 \text{ in } \Omega,\; u = z \text{ on } \partial^m\Omega$ where z is a given function on $\partial^m\Omega$. Then $\exists\, K, \kappa > 0$, Ω-independent, such that*

$$|u_\xi| \le K\, e^{-\kappa\, d(\xi, \text{support of z})}\, |z|_\infty$$

Furthermore the following theorem (almost obvious in this case) expresses the local support property at $\xi \in \Omega$:

$$\mu_A(\{\varphi \,|\, |\varphi_\xi| > B\}) \le c_7 e^{-c_8 B^2}$$

valid also for μ_A^O associated with the operator D with Dirichlet (*i.e.* null) boundary c. on $\partial^m\Omega$, and for all $\Omega \in Z^d$).

$$\mu_A^\Omega(\{\varphi\,|\,|\varphi_\xi| > B\}) \le c_7 e^{-c_8 B^2} \qquad \forall\,\xi$$

if c_7, c_8, Ω–independent, are suitably chosen.
Finally use the following *Markov property*:

Given $\Omega \subset \Lambda$ the field φ with distribution μ_A can be represented as a sum of two independent fields $\varphi = u + \zeta$ where:
(i) ζ is the Gaussian field on Ω with Dirichlet (null) b. c. outside Ω (call μ_A^Ω the distribution of ζ, $\zeta_\xi = 0$ if $\xi \notin \Omega$).
(ii) u for $\xi \notin \Omega$ is distributed as the field with covariance A (i.e. $u_\xi = \varphi_\xi$, $\xi \notin \Omega$) and inside Ω it is determined by its values outside Ω as the solution of

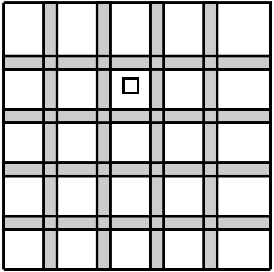
$$(1-D)^m u = 0 \quad \text{in } \Omega, \qquad u_\xi = \varphi_\xi \quad \text{in } \partial^m\Omega$$

provided $\mathrm{diameter}(\Omega) < \frac{1}{2}\mathrm{diameter}(\Lambda)$.

This means that if $F(\varphi)$ depends only on φ outside Ω and G^4 is a function of φ depending only on its values in Ω then

$$\int F(\varphi)G(\varphi)\mu_A(d\varphi) = \int F(u)G(u+\zeta)\mu_A(d\varphi)\mu_A^{\Omega}(d\zeta)$$

Let, $\forall\,\Gamma \subset \Lambda$**:** $\chi_{\Gamma}^{B}(\varphi) = 1\big(\text{if } \{|\varphi_\xi| < B,\ \forall \xi \in \Gamma\}\big)$ **and let**



$$P(B) = \int \mu_A(d\varphi)\,\chi_{\Lambda}^{B}(\varphi).$$

Divide Λ into boxes with side size $R \ll \Lambda$, denoted as $\square$ leaving a corridor $\mathcal{C}$ of width m Now integrate over φ as follows

$$P(B) = \int \chi_{C}^{B}(\varphi) \prod_{\square} \chi_{\square}^{B}(\varphi)\mu_A(d\varphi) =$$
$$= \int \chi_{C}^{B}(\varphi)\mu(d\varphi) \prod_{\square} \Big(\int \chi_{\square}^{B}(u+\zeta)\Big)\mu_A^{\mathcal{C}}(d\zeta)$$

Having used the Markov property.

The elliptic problem implies $\exists \gamma, \kappa > 0$, R, L independent, s.t.:

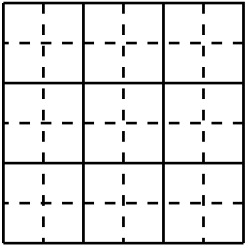
$$|u_\xi| \le \gamma \max_{\xi \in \mathcal{C}} |\zeta_\xi| \le \gamma\, B, \quad \gamma = \sum_{\zeta_1,\ldots,\zeta_d, \zeta_d \ge 0} 2^d K e^{-\frac{1}{2}\kappa|\zeta|} < \infty, \ \kappa, K > 1$$

Hence $\chi^B_\xi(u+\zeta) \ge \chi^{B/2}_\xi(u)\, \chi^{B/2}_\xi(\zeta)$ **and**

$$P(B) \ge \int \chi^{B/2\gamma}_{\mathcal{C}}(\varphi) \prod_{\square} \Big(\int \chi^{B/2}_{\square}(\zeta) \mu^{\mathcal{C}}_A(d\zeta) \Big) \mu_A(d\varphi) \ge$$

$$\ge \exp\Big(- c_{10} e^{-c_{11} B^2} \frac{|\Lambda|}{|\square|} \Big) \int \chi^{B/2\gamma}_{\mathcal{C}}(\varphi) \mu_A(d\varphi)$$

as ζ fields are independent (satisfy Dirichlet b,c.)



We can repeat the argument to estimate last integral $P'(B)$. Consider again a pavement of Λ by cubes of the previously used size (with corridors $\mathcal{C}'$) but with centers shifted along the diagonal of the old ones by $\frac{1}{2}$-diagonal.

Call d_ξ the distance of ξ from the *old* corridors $\mathcal{C}$:

Note that since we use periodic boundary conditions the set of shifted boxes still covers the box Λ.

$$
\begin{aligned}
P'(B) &= \int \chi_{\mathcal{C}}^{B/2\gamma}(\varphi)\mu_A(d\varphi) \geq \\
&\geq \int \chi_{\mathcal{C}}^{B/2\gamma}(\varphi) \prod_{\xi\in\mathcal{C}'} \chi_\xi^{Be^{\kappa d_\xi/2}/(2\gamma)^2}(\varphi)\mu_A(d\varphi) \geq \\
&\geq \Big(\int \mu_A(d\varphi) \prod_{\xi\in\mathcal{C}'} \chi_\xi^{B/(2\gamma)^2 e^{\kappa/2 d_\xi}}(\varphi)\Big) \exp -\frac{|\Lambda|}{|\square|} c_{12} e^{-c_{13}B^2}
\end{aligned}
$$

First inequality implied by the extra characteristic functions; then use Markov property with respect to the new corridors $\mathcal{C}'$ and that $|\varphi| = |u+\zeta| < \frac{B}{2\gamma}$ due to the stronger conditions **(i)** $|\varphi_\xi| \leq \frac{B}{(2\gamma)^2} e^{\kappa d_\xi/2}$ in the corridor $\mathcal{C}'$ (implying that u is everywhere $\leq B/4\gamma$) and
(ii) $|\zeta| < B/4\gamma$ in each of the boxes $\square$ into which the new pavement divides Λ: event probability bounded by $\exp -c_{12}e^{-c_1 B^2}$ per box $\square$.

Then make a new identical argument by shifting the pavement along the diagonal by $\frac{1}{4}$ the diagonal. The last integral, denote it $P''(B)$, is bounded below by

$$P''(B) \geq \int \Big(\prod_{\xi \in \mathcal{C}''} \chi_\xi^{(B/(2\gamma)^3)e^{\kappa/2\,(d_\xi + d_{\xi'})}}(\varphi)\mu_A(d\varphi)\ \exp - \frac{|\Lambda|}{|\square|} c_{14} e^{-c_{15} B^2} \Big)$$

$$\geq \int \Big(\chi_{\mathcal{C}''}^{Be^{\frac{\kappa}{2}\frac{R}{4}}/(2\gamma)^3}(\varphi)\, d\mu_A \Big) \exp - \frac{|\Lambda}{|\square|} c_{14} e^{-c_{15} B^2}$$

$$\geq e^{-\frac{|\Lambda|}{|\square|} c_{14} e^{-c_{15} B^2}} P\big(\frac{Be^{\frac{1}{8}\kappa R}}{(2\gamma)^3}\big)$$

Collect inequalities and use arbitrariness of R s.t. $G \stackrel{def}{=} \frac{e^{\frac{1}{8}\kappa R}}{(2\gamma)^3} > 2$, it is $P(B) \geq \big(e^{-c_{16} e^{-c_{17} B^2} |\Lambda|}\big)\, P(2B)$ Since $P(B) \xrightarrow[B \to \infty]{} 1$, by iteration:

$$P(B) \geq \exp - |\Lambda| c_{16} \sum_{u=0}^{\infty} e^{-c_{17} 2^{2n} B^2} \geq \exp - |\Lambda| c_{18} e^{-c_{19} B^2}$$

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Quoted references

[1] G. Benfatto, G. Gallavotti, and F. Nicoló.
Elliptic equations and gaussian processes.
Journal of Functional Analysis, 36:343–400, 1980.