Let $Q_i, i = 0, 1, \ldots$ be a sequence of compatible pavements of R^d with sides $2^{-i}, i = 0, 1, \ldots$.

In $\Delta \in Q_i$ is defined a variable $z_\Delta(x)$ and $E(z_\Delta(x)^2) = \frac{1}{2}$

Field at a point x is $\varphi_{\mathbf{x}} = \sum_{\mathbf{i=0}}^{\mathbf{N}} \mathbf{z_{\Delta_i}(x) 2^{\frac{1}{2}(d-2)i}}$. Models:

0) Ising h. fields: $z_\Delta(x) = z_\Delta = \pm 1$, x-indep. in each Δ:
$P(d\mathbf{z}) = \prod_\Delta \sum_{z_D = \pm 1}$

1) Bernoulli h. fields: $z_\Delta(x) = z_\Delta = \pm 1$, x-indep. in each Δ:
$P(d\mathbf{z}) = \prod_\Delta \frac{e^{-\frac{1}{2} z_\Delta^2} dz_\Delta}{2\pi}$

2) Markov h. fields: $z_\Delta(x)$ x-indep. in each Δ

$$Z_N^{-1} \exp -\frac{\beta}{2} \Big[\sum_{\Delta, \Delta' \in Q_i}^{N\,*} (z_\Delta - z_{\Delta'})^2 + \alpha^2 \sum_{\Delta \in Q_i} z_\Delta^2 \Big]$$

$\sum^*$ over i & n.n. $\Delta, \Delta' \in Q_i$; $\beta, \alpha > 0$ fixed s.t. $\langle z_\Delta^2 \rangle$ is $\frac{1}{2}$.

(3) Euclidean h. fields: Gaussian with covariance $C^{\leq N}_{x,y} = \sum_{i=0}^{N} 2^{(d-2)i} C^{0}_{2^i x, 2^i y}$, *i.e.* $C^{\leq N}_{k} = \frac{1}{1+k^2} - \frac{1}{2^N + k^2}$

$$C^{(0)}_{x,y} = \frac{1}{(2\pi)^d} \int \frac{3}{(1+k^2)(4+k^2)} e^{ik(x-y)} d^d k$$

In all cases $\varphi^{(\leq N)}_x$ **can be large** $\langle (\varphi^{(\leq N)}_x)^2 \rangle = c \sum_{i=0}^{N} 2^{(d-2)i}$.

In all cases $\varphi^i_x = c 2^{\frac{1}{2}(d-2)i} \frac{z^i_x + \sqrt{\Gamma_i} X^{i-1}_x}{\sqrt{1+\Gamma_i}}$, $\Gamma_i = \sum_{n=0}^{i-1} \frac{2^{(d-2)n}}{2^{(d-2)i}}$, **and** z^i, X^i **are constant in each** $\Delta \in Q_i$ **(in models 1,2) or almost constant in model 3,** *i.e.* **are** ε**-Hölder c.**

Why Hölder: let $w = \frac{z^0_x - z^0_y}{|x-y|^\varepsilon} = \int \widehat{z}^0_k \frac{(e^{ikx} - e^{iky})}{|x-y|^\varepsilon} d^d k$**; hence** $\langle w^2 \rangle = \int \frac{3}{(1+k^2)(4+k^2)} k^{2\varepsilon} d^d k = c < \infty$ **implies, if** $2\varepsilon + d < 4$ **and** x, y **are in the same** $\Delta \in Q_0$ **(or 'close'):**

$$\varepsilon < 1, \text{ if } d = 2 \qquad \varepsilon < \frac{1}{2} \text{ if } d = 3$$

In all cases z^i, z^j are independently distributed for $i \neq j$ and at each scale correlation decays exponentially on the respective scale, if present at all (in model 1 is absent).

Detailed UV-analysis deals with upper and lower bounds for $\int e^{\int_I :(\varphi_x^{\leq N})^4: d^d x}$, where I is the unit box. And it relies on

Lemma: *Let Λ be a cube and $\Lambda_1, \Lambda_2, \ldots$ be unit cubes paving Λ. Let $\alpha_1 \in C^\infty(R^d)$, with support in Λ_1 and let $\alpha_j(x) = \alpha_1(x - \xi_j)$ where ξ_j is the center of Λ_j. Then if $s < 4 - \frac{d}{2}$, $\exists\, c_1, c_2$, Λ-independent constants, such that:*

$$\mu_A(\varphi \,|\, max_j \|\alpha_j \varphi\|_{H_s(R^d)} \leq B) \geq e^{-c_1 e^{-c_2 B^2} |\Lambda|}$$

and if s=integer, $s + \varepsilon < 2 - \frac{d}{2}$ the probability that the Hölder C_ε norm $\|\alpha_j \varphi\|_{C_\varepsilon} < B$ is

$$\mu_A(\max_j \|\alpha_j \varphi\|_{C_\varepsilon} < B, \forall j) \geq e^{-c_1 e^{-c_2 B^2 |\Lambda|}}$$

All examples have the properties:
a) almost independently valued on each scale h,
b) almost constant at scale h, by the lemma above
c) for all $\Delta \in Q_N$ (up to large fluctuations):

$$V_\Delta = \int_\Delta \lambda : (\varphi_x^{\le N})^4 : 2^{-dN} dx = O(\lambda 2^{4\frac{(d-2)}{2}} 2^{-dN}) = \mathbf{0}(\lambda \mathbf{2}^{(\mathbf{d}-\mathbf{4})\mathbf{N}})$$

Therefore suggestion is to study, if $d = 2, 3$:

$$\int e^{\lambda \int_I :(\varphi_x^{\le N})^4: dx} = \int \mathbf{e}^{-\lambda \sum_{\Delta \in Q_N} 2^{(d-4)N} H_4\left(\frac{z_\Delta^N + \sqrt{\Gamma_N} X_\Delta^{N-1}}{\sqrt{1+\Gamma_N}}\right)}$$

simply by a careful perturbative evaluation of the integral over z_Δ distinguishing the cases in which $|z_\Delta|, |X_\Delta|$ are respectively above or below a treshold

$$B_\Delta = B(1+N)^p (\log(e + \lambda^{-1}))^2, \Delta \in Q^N$$

for $p = 4, 2$.

The procedure is the same in the above models: in Euclidean case Hölder continuity lemma essentially reduces the problem to the Markov or even Bernoulli models.

The lemma was proved in [1] and is a legacy of Benfatto, whose contribution was essential as well as the idea to introduce and use it in the work on the Markov case [2].

The $d = 2$ Bernoulli field contains the key ideas:

$$\varphi_x^N = \sum_{\Delta \ni x, \Delta \in \overline{Q}_i} \sqrt{2\gamma_i} z_\Delta^N, \ \gamma_i = 1, \ \mathcal{E}((\varphi_x^N)^2) = \sum_{\substack{\Delta \ni x \\ \Delta \in \overline{Q}_i}} 1 = N + 1$$

$$\varphi_{\mathbf{x}}^{\mathbf{N}} = \sqrt{\mathbf{N}+\mathbf{1}} \mathbf{X}_{\mathbf{\Delta}}^{\mathbf{N}} = \sqrt{\mathbf{N}+\mathbf{1}} \, \frac{\mathbf{z}_{\mathbf{\Delta}}^{\mathbf{N}} + \sqrt{\mathbf{N}} \mathbf{X}_{\mathbf{\Delta'}}^{\mathbf{N}-\mathbf{1}}}{\sqrt{\mathbf{1}+\mathbf{N}}}, \ \int \mathbf{e}^{\lambda \int_{\mathbf{I}} :(\varphi_{\mathbf{x}}^{\mathbf{N}})^4: \, \mathbf{dx}}$$

$$:(\varphi_{\mathbf{x}}^{\mathbf{N}})^4 := (\mathbf{1}+\mathbf{N})^2 \mathbf{H}_4\left(\frac{\mathbf{z}_{\mathbf{\Delta}}^{\mathbf{N}} + \sqrt{\mathbf{N}} \mathbf{X}_{\mathbf{\Delta'}}^{\mathbf{N}-\mathbf{1}}}{\sqrt{\mathbf{1}+\mathbf{N}}}\right)$$

$$\mathbf{V_l^N} = \lambda \sum_{\mathbf{\Delta \in Q_N}} (\mathbf{1{+}N})^2 \mathbf{H_4}\Big(\frac{\mathbf{z_\Delta} + \sqrt{\mathbf{N}}\mathbf{X_{\Delta'}}}{\sqrt{\mathbf{1+N}}}\Big)\, \sigma_{\mathbf{N}}, \qquad \sigma_{\mathbf{N}} = |\mathbf{\Delta}| = \mathbf{2^{-2N}}$$

'Small fields' integration:

$$\int \mathbf{e^{V_l^N}} \chi^{\mathbf{N}}, \qquad \chi^{\mathbf{N}} = \prod_{\mathbf{k=0}}^{\mathbf{N}} \prod_{\mathbf{\Delta \in Q_k}} \chi_{\mathbf{\Delta}}(|\mathbf{X_\Delta^N}| < \mathbf{B_k})$$

$$\mathbf{B_k} = (\mathbf{1+k})^{\mathbf{4}}\mathbf{B_0},\ \ \mathbf{B_0} = (\mathbf{1} + \log\lambda)^{\mathbf{2}}\mathbf{B},\ \ \mathbf{B_0} > \mathbf{3}$$

Recursively: $\widetilde{\mathbf{V}}_{\mathbf{l}}^{\mathbf{N}} = \mathbf{V_l^N},\ \ \widetilde{\mathbf{V}}_{\mathbf{l}}^{(\mathbf{N-k})} = \mathcal{E}_{\mathbf{N-k+1}}(\mathbf{V_l^{(N-k+1)}}),\ \ \widetilde{V}_l^{-1} = 0$

$$\int e^{V_l^N} P^N(dX^N) \geq \int e^{V_l^N} \chi^N P^N(dX^N) = \int \chi^{N-1} P^{N-1}(dX^{N-1}).$$

$$\prod_{\mathbf{\Delta \in Q_N}} \Big(\int \chi\Big(\Big|\frac{\mathbf{z_\Delta} + \sqrt{\mathbf{N}}\mathbf{X_{\Delta'}^{N-1}}}{\sqrt{\mathbf{N+1}}}\Big| < \mathbf{B_N}\Big) \mathbf{e}^{-\lambda\sigma_{\mathbf{N}}(\mathbf{1+N})^{\mathbf{2}}\mathbf{H_4}\left(\frac{\mathbf{z_\Delta} + \sqrt{\mathbf{N}}\mathbf{X_{\Delta'}}}{\sqrt{\mathbf{N+1}}}\right)} \frac{\mathbf{e}^{-\frac{\mathbf{z_\Delta^2}}{\mathbf{2}}}\mathbf{dz_\Delta}}{\sqrt{\pi}}\Big)$$

Integrals over z_Δ's *independent*: single Δ is computed $O(\lambda^2)$:

$$\mathbf{1}-\lambda\sigma_{\mathbf{N}}(\mathbf{1}+\mathbf{N})^{\mathbf{2}}\int \mathbf{H_4}\Big(\frac{\mathbf{z}+\sqrt{\mathbf{N}}\mathbf{X}_{\mathbf{\Delta'}}}{\sqrt{\mathbf{N}+\mathbf{1}}}\Big)\frac{\mathbf{e}^{-\mathbf{z}^2}\mathbf{dz}}{\sqrt{\pi}}$$

$$+\mathbf{O}(\lambda^{\mathbf{2}}\sigma_{\mathbf{N}}^{\mathbf{2}}\mathbf{B_0^8}\mathbf{N}^{\mathbf{18}}\mathbf{e}^{\frac{3}{2}\lambda\mathbf{N}^2\sigma_{\mathbf{N}}})$$

$$=\mathbf{1}-\lambda\sigma_{\mathbf{N}}\mathbf{N}^{\mathbf{2}}\mathbf{H_4}(\mathbf{X}_{\mathbf{\Delta'}})+\mathbf{O}(\lambda^{\mathbf{2}}(\log\lambda)^{\mathbf{8}}\mathbf{e}^{\mathbf{c}\lambda}\mathbf{N}^{\mathbf{20}}\sigma_{\mathbf{N}}^{\mathbf{2}})$$

$$=\mathbf{e}^{-\lambda\sigma_{\mathbf{N}}\mathbf{N}^{\mathbf{2}}\mathbf{H_4}(\mathbf{X}_{\mathbf{\Delta'}})+\mathbf{O}(\lambda^{\mathbf{2}}(\log\lambda)^{\mathbf{2}}\mathbf{e}^{\mathbf{c}\lambda}\mathbf{N}^{\mathbf{20}}\sigma_{\mathbf{N}}^{\mathbf{2}})}$$

By $4\sigma_{\mathbf{N}}\mathbf{N}^2=\sigma_{\mathbf{N}-\mathbf{1}}(\mathbf{1}+(\mathbf{N}-\mathbf{1}))^2$, and $V^{-1}=0$, recompose as

$$\int e^{V_I^N}\chi^N P(dz^N)\geq e^{V_I^{N-1}}e^{\lambda^2(\log\lambda)^2e^{c\lambda}N^{20}\sigma_N^2\,2^{2N}I}$$

$$\geq e^{-\lambda^2\varepsilon(N,\lambda)\sigma_N^2 2^{2N}|I|}$$

i.e. **each $\Delta\in Q_N$ contributes $\lambda^2\varepsilon(N,\lambda)\sigma_N^2$,**
$\varepsilon(N,\lambda)=(\log\lambda)^2e^{c\lambda}N^{20}$**; hence, by $\sigma_N^2 2^{2N}=2^{-2N}$:**

$$\int e^{V_I^N}\prod_{i=0}^{N}\chi^i(z^i)P(dz^i)\geq e^{-\sum_{i=0}^N\lambda^2\varepsilon(i,\lambda)2^{-2i}|I|}$$

Now upper bound in view of the application to Markov fields.

Let $\chi^i_D=\prod_{\Delta\subset D}\chi(|X^i_\Delta|>B_i), \widehat\chi^i_D=\prod_{\Delta\subset D}\chi(|X^i_\Delta|<B_i)$ **s.t.**
$\sum_D e^{V^N}\chi^N_D\widehat\chi^N_{D^c}\le\sum_D e^{V^N_{D^c}}\chi^N_D\widehat\chi^N_{D^c}$ **or even, integrating:**

$$\le\sum_{\mathbf{D^{N-1}}}\sum_{\mathbf{D}}\int \mathbf{e}^{\mathbf{V^N_{D^c}}+\mathbf{V^{N-1}_{D/D_{N-1}}}}\chi^{\mathbf{N}}_{\mathbf{D}}\widehat\chi^{\mathbf{N}}_{\mathbf{D^c}}\chi^{\mathbf{N-1}}_{\mathbf{D^{N-1}}}\widehat\chi^{\mathbf{N-1}}_{\mathbf{D^{N-1,c}}}$$

because $|X_\Delta|>B_N\gg 3$ **implies** $V_\Delta(X_\Delta)\gg 0$ **and** X_Δ **is so large in** D/D_{N-1} **that, even replacing** $V^N(D/D_{N-1})$ **with** $V^{N-1}(D/D_{N-1})$**, yields an upper bound.**

By the lower bound analysis

$$\sum_D\int e^{V^N_{D^c}}\widehat\chi^N_{D^c}\chi^N_D\le\sum_{D^{N-1}}\int e^{V^{N-1}_{D^{N-1,c}}+\varepsilon(N,\lambda)2^{-2N}|I|}\chi^{N-1}_{D^{N-1}}\widehat\chi^{N-1}_{D^{N-1,c}}$$

as:

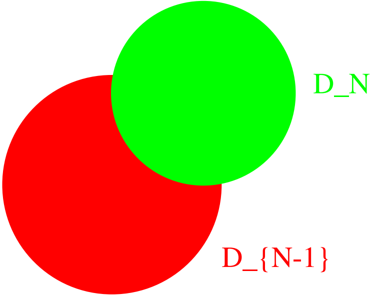

Figure: **Green=large** $|X^N| > B_N$**, Red=large** $|X^{N-1}| > B_{N-1}$**.**

Replace $V^N_{D^c_N}$ **with** $V^{N-1}_{D^c_N} + V^{N-1}_{D_{N-1}/D_N} = V^{N-1}_{D^c_{N-1}}$**: here enters the positivity of** $H_4(x)$ **for** $x > 2$**.**

$$\int \mathbf{e}^{\mathbf{V^N_I}} \chi^{\mathbf{N}}(\mathbf{D}) \widehat{\chi}^{\mathbf{N}}(\mathbf{D^c}) \leq \mathbf{e}^{\mathbf{V^{N-1}_I} + \lambda^2 \varepsilon(\mathbf{N}, \lambda) \mathbf{2}^{-\mathbf{2N}} |\mathbf{I}|} \chi^{\mathbf{N-1}}(\mathbf{D})$$

so that the $E_+ = \lambda^2 \sum_N \varepsilon(N, \lambda) 2^{-2N} |I|$ **is an upper bound.**

Analysis to higher order: no extra work needed for

$$\mathbf{E}_{\pm} = \sum_{\mathbf{k}=2}^{\mathbf{t}} \lambda^{\mathbf{t}} \mathbf{S}_{\mathbf{t}} |\mathbf{I}| \pm \lambda^{\mathbf{t}+1} \sum_{\mathbf{N}} \varepsilon_{\mathbf{t}}(\mathbf{N}, \lambda) \mathbf{2}^{-\mathbf{N}} |\mathbf{I}|$$

where S_t are coefficients predicted by a perturbation analysis (finite to all orders as the theory is "superrenormalizable") and $\varepsilon_t(N, \lambda) = (\log \lambda)^2 e^{c_t \lambda} N^{20t}$;

In Markov models explicit integration over $z_\Delta, \Delta \in Q_N$ can be replaced by the cluster expansion or by an argument of probability theory to control exponentially decaying correlations z_Δ, z'_Δ,

[[3] with cluster expansion or [4] with a more standard probabilistic method].

The lower bound amounts to estimating, in presence of the $\chi^N = \{\varphi || z_\Delta + X_{\Delta'}| < B_N|, |X_{\Delta'}| < B_{N-1}\}$ to a finite order in λ: $W(I) = \int \chi^N e^{\sum_{\Delta\in Q_N} \lambda\sigma_N H_4(\frac{z_\Delta + X_{\Delta'}}{2})} = \int \prod_\Delta w(\Delta)$. Then:

$$W(I) = \int \prod_{\Delta\in Q_N} \{(w(\Delta)-1)+1\} = \sum_{\mathbf{m}} \sum_{\mathbf{R_1,R_2,\ldots,R_m}} \mathbf{K(R_1)\ldots K(R_m)} \qquad (*)$$

where R_j is a collection of $\Delta' s$ forming a connected set R_j, and $R_i \cap R_j = \emptyset$ and

$$|K(R)| \le (\lambda\sigma_N e^{O(\lambda N^{20}\sigma_N)})^{|R|}$$

This is a "short range hard core interaction" between polymers R_i and a typical quantity to compute is partition function to show validity of its perturbation expansion.

In Euclidean models the Hölder continuity on scale N Lemma, allows treating the problem as in Markov cases, [2].

The Ising hierarchy ($z_x^N = z_\Delta^N = \pm 1, \forall x \in \Delta \in Q^N$):

$$\varphi_x^{\le N} = (1+N)\frac{z_x^N + N X_x^{N-1}}{\sqrt{1+N}}, \quad d = 2$$

$$\varphi_x^{\le N} = 2^N \frac{z_x^N + X_x^{N-1}}{2}, \quad d = 3$$

is an easy warm-up exercise (at least in the $d = 2$ case)

The Euclidean sine-gordon model can be treated along the same lines and it is somewhat simpler, [5].

Quoted references

G. Benfatto, G. Gallavotti, and F. Nicoló.
Elliptic equations and gaussian processes.
Journal of Functional Analysis, 36:343–400, 1980.

G. Benfatto, M. Cassandro, G. Gallavotti, F. Nicolò, E. Olivieri, E. Presutti, and E. Scacciatelli.
Ultraviolet stability in euclidean scalar field theories.
Communications in Mathematical Physics, 71:95–130, 1980.

G. Gallavotti.
On the ultraviolet stability in Statistical Mechanics and field theory.
Annali di Matematica pura ed applicata, CXX:1–23, 1979.

G. Benfatto, M. Cassandro, G. Gallavotti, F. Nicolò, E. Olivieri, E. Presutti, and E. Scacciatelli.
Some probabilistic techniques in field theory.
Communications in Mathematical Physics, 59:143–166, 1978.

G. Gallavotti.
Renormalization theory and ultraviolet stability for scalar fields via renormalization group methods.
Reviews of Modern Physics, 57:471–562, 1985.

Proceed to the upper bound starting from the identity:

$$\mathbf{1} = \sum_{\substack{\Delta_i \subset Q_i \\ i=0,\ldots,N}} \prod_{i=0}^{N} \left\{ \prod_{\Delta \in D_i} \chi(|\mathbf{X}_\Delta| > \mathbf{B}_i) \prod_{\Delta \notin D_i} \chi(|\mathbf{X}_\Delta| < \mathbf{B}_i) \right\}$$

and set: $\mathbf{V}(\mathbf{X}_\Delta) = \mathbf{0}, \boldsymbol{\Delta} \in \mathbf{Q}_N,$ if $|\mathbf{X}_\Delta| > \mathbf{B}_N$ **(as** $H_4(x) > c\,x^4,\ if\ x > 3$**). The integral to compute is then**

$$\sum_{\mathbf{D}_N \subset \mathbf{Q}_N} \int e^{-\lambda \sigma_N \sum_{\Delta \in D_N} H_4(\frac{z_\Delta + \sqrt{N} X_{\Delta'}}{\sqrt{1+N}})} \prod_{\Delta \in D_N} \chi(|\frac{z_\Delta + \sqrt{N} X_{\Delta'}|}{\sqrt{N}} < B_N)$$

$$\prod_{\Delta \notin \mathbf{D}_N} \chi(|\frac{\mathbf{z}_\Delta + \sqrt{\mathbf{N}} \mathbf{X}_{\Delta'}|}{\sqrt{\mathbf{N}}} > \mathbf{B}_N)\ \frac{\mathbf{e}^{-\mathbf{z}_\Delta^2} \mathbf{d}\mathbf{z}_\Delta}{\sqrt{\pi}}$$

each $\Delta' \in Q_{N-1}$ **contains** $\mathbf{j}_{D'} = \mathbf{0}, \mathbf{1}, \ldots, \mathbf{4}$ $\Delta \in D_N$ **with** $|\frac{\mathbf{z}_\Delta + \sqrt{\mathbf{N}} \mathbf{X}'_\Delta|}{\sqrt{\mathbf{N}}} < \mathbf{B}_N)$**. Let** $D' \subset Q_{N-1}$ **such that each cube in** D' **contains at least one** $\Delta \in D_N$ **with** $j_\Delta < 4$ **(***i.e.* **a** Δ **with** $|X_\Delta| > B_N$**).**

Hence the integral is, if $\{j_\Delta\}_{\Delta\in D_N}$

$$P(B_N, D')\prod_{\Delta'\in D'}(e^{j_{\Delta'}\frac{1}{4}E_-(\lambda,B_N)+\lambda\sigma_N^2 B_N^8 e^{c\lambda}})$$

where $P(B_N, D')$ **= probability that** $j_\Delta > 0, \forall\Delta \in D'$**.**

By the lower bound the product is $\le e^{+\lambda^2(\log\lambda)^2\varepsilon(N,B_N)2^{-N}|I|}$**.**

The number of $\Delta \in Q^{n-1}$ **labeled by** $j_\Delta > 0$ **is** $4^{2^{2i}}$ **s. t.:**

$$\sum_{D'} P(B_N, D') \le \sum_{\neq D'\subset Q_{N-1}} e^{c-B_N^2|D'|} \le (1+e^{c-B_N^2})^{2^2 N}|I|$$

Iterating over N **an upper bound is**

$$+E_-(\lambda)|I| + \sum_{N=0}^{\infty} e^{(c-B_N^2)2^{2N}|I|} = +E_-(\lambda)I = O(\lambda^\infty)$$

(because $B_N = (1+N)^4(\log\lambda)^2 B_0$**).**