Friction and irreversibility in equations on macroscopically irreversible phenomena

Viscosity is a property attributed to exchange of momentum across an ideal plane: Maxwell (1860) derived the viscosity coefficient ν in a hard balls gas and later on a gas of pairwise interacting molecules (1866).

Remarkably derives ν studying collisions effects via a "weak" Boltzmann equation, (perhaps explaining also why in his celebration of Boltzmann's work accent is on the ensembles while B.E. is not mentioned).

Viscosity emerges as due to the stochastic molecular collisions: *i.e.* it is clear that it is an average property of macroscopicall observed ymotions.

Taking it into account introduces phenomenologic forces which break the time reversal symmetry which is, however, a fundamental symmetry.

And a natural question is whether alternative theories may provide the effects of friction while preserving time reversal (avoiding obvious return to the microscopic dynamics).

It is wise to fix attention to a special case but aiming at more general cases.

Here I select the example of the Navier-Stokes incompressible and in a container with periodic boundary $[0,2\pi]^d$, $d=2,3$.

$$\dot{\mathbf{u}}(x) = -(\underset{\sim}{\mathbf{u}}\cdot\underset{\sim}{\partial})\mathbf{u} + \nu\Delta\mathbf{u} + \mathbf{f} - \partial p, \qquad \boldsymbol{\partial}\cdot\mathbf{u} = 0 \qquad NS$$

$\mathbf{u}(x)$ **velocity field,** $\mathbf{f}(x)$ **fixed forcing,** p **pressure,** ν **viscosity.**

The equation has a rather complete mathematical status if $d=2$ **while in** $d=3$ **it is, to say the least, not yet satisfactory.**

This is not unusual: for instance other, even more fundamental, equations are not fully undestood.

Like the equations of motion of infinite systems of particles: rather than inducing to avoid their analysis, existence-uniqueness of solutions is assumed, and consequences are derived, although only in very special cases established and in spite of the relevance for Stat. Mech.

For instance the problem of the existence of the dynamics is often bypassed by imagining the systems enclosed in large but finite containers: while this avoids the existemce-uniqueness parts of the problems, it leaves open essentially all the ones that are of interest.

On NS: develop velociy in complex $u_{\mathbf{k},c} = \overline{u}_{-\mathbf{k},c}$ harmonics:

$$\mathbf{u}(x) = \frac{1}{(2\pi)^d} \sum_{\mathbf{k},c} i e^{i\mathbf{k}\cdot\mathbf{x}} u_{\mathbf{k},c}(x)\, \varepsilon^c_{\mathbf{k}}, \qquad \mathbf{k} \in Z^d,\ c = 1, ..$$

$\varepsilon^c_{\mathbf{k}} = -\varepsilon^c_{-\mathbf{k}}$ **unit vectors orthogonal to k:** *helicities*.

$$\dot{\mathbf{u}} = -\mathbf{u}\cdot\partial\mathbf{u} + \nu\Delta\mathbf{u} - \partial p + \mathbf{f}$$

UV-N Regularization: this is the analog of the reduction to a finite container of the particle systems, which disposes of the existence-uniqueness of motions of infinite gases. It consists in setting $u_{\mathbf{k},c} = 0$ for all $|\mathbf{k}| \le N$.
The equations in Fourier's transform are

$$\dot{u}_{\mathbf{k},c} = -\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k};a,b} T^{a,b,c}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}} u_{\mathbf{k}_1,a} u_{\mathbf{k}_2,b} - \nu\mathbf{k}^2 u_{\mathbf{k},c} + f_{\mathbf{k},c}$$

$$T^{a,b,c}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}} = (\varepsilon^a_{\mathbf{k}_1}\cdot\mathbf{k}_2)(\varepsilon^b_{\mathbf{k}_2}\cdot\varepsilon^c_{\mathbf{k}})$$

What can be said at least about the stationary distributions if the forcing acts on "large scale", *i.e.* **$f_{\mathbf{k}}$ is not zero only for a few small k, say finitely many.**

In $d = 2$ energy and enstrophy $E = \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2$, $\mathcal{D} = \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 |\mathbf{k}|^2$ are bounded proportionally to ν^{-2} uniformly in N; $d = 3$??.

In the literature the NS has been studied in simulations and a **fairly wide agreement** focuses on the stationary statistical properties of $\mathbf{u}_k$ for $|\mathbf{k}| < k_c = LR^{\frac{1}{2}}$ if $d = 2$ and $LR^{\frac{3}{4}}$ and if $d = 3$: here R is Reynolds' num. of the stationary flow. $R = Lv\nu^{-1} = 2\pi E^{\frac{1}{2}}\nu^{-1}$.

About friction: **since** the basic phenomena are reversible it is possible that the very same fluid can be described by reversible equations **even macroscopically**.

The suggestion is that the properties of a fluid could be studied by other equations: as in equilibrium very different evolutions produce the same average results on large classes of observables; think of Isoenergetic evolution ($E = const$, leading to the microcanonical ensemble) or Isokinetic evolution ($K = const.$)

In the following it is argued that a different equation is physically equivalent and even reversible.

The new equation is simply $\nu \to \alpha(\mathbf{u}) = \frac{\mathbf{f}\cdot\Delta\mathbf{u}+(\mathbf{u}\cdot\boldsymbol{\partial}\mathbf{u})\cdot\Delta\mathbf{u}}{\mathbf{u}\cdot\Delta^2\mathbf{u}}$

$$\dot{u}_{\mathbf{k},c} = -\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k};a,b} T^{a,b,c}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}} u_{\mathbf{k}_1,a} u_{\mathbf{k}_2,b} - \boldsymbol{\alpha}\mathbf{k}^2\, u_{\mathbf{k},c} + f_{\mathbf{k},c}$$

with α such that the Enstrophy $\mathcal{D} = \sum_{\mathbf{k},c} |\mathbf{k}^2 u^c_{\mathbf{k}}|^2$ is an exact constant of motion. This means

$$\alpha(\mathbf{u}) = \frac{\sum_{\mathbf{k},c} f_{\mathbf{k},c}\mathbf{k}^2 u_{\mathbf{k},c} + \sum^{a,b,c}_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}=0} T^{a,b,c}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}}\, \mathbf{k}^2\, u^a_{\mathbf{k}_1} u^b_{\mathbf{k}_2} u^c_{\mathbf{k}}}{\sum_{\mathbf{k},c} \mathbf{k}^4 |u^c_{\mathbf{k}}|^2}$$

This new equation "RNS"subject to UV-N regularization is as difficult as NS. The conjecture is that it is equivalent in the limit $N \to \infty$ to the viscous NS.

The new eq. conserves enstrophy $\mathcal{D} = \sum_{\mathbf{k}} \mathbf{k}^2 |u_{\mathbf{k}}|^2$ and is time-reversible, hence "RNS".

Call *En* average of $\mathcal{D}$ in the stationary state of NS and $\overline{\alpha}$ average of α in the reversible equation RNS

Correspondent states: are stationary distributions μ_ν^N distrib. for the NS eq. and $\widetilde\mu_{En}^N$ for the RNS if:

$$\overline{\alpha} \stackrel{def}{=} \widetilde\mu_{En}^N(\alpha) = \nu \qquad \textit{or} \quad \mu_\nu^N(\mathcal{D}) = En$$

Correspondent means *equal average dissipation*.

Conjecture: If given En and ν the states $\widetilde\mu_{En}^N$ and μ_ν^N are correspondent then for all local observables F:

$$\lim_{N\to\infty} \mu_\nu^N(F) = \lim_{N\to\infty} \widetilde\mu_{En}^N(F)$$

Local means that the observable $O(u)$ depends only on finitely many components $u_{\mathbf{k}}$, (or on $\mathbf{k} < k_c$=Kolmogorov's scale)

Remarks: *Similarity with the thermod. limit canonical–grand_canonical equivalence. As in Stat.Mech. there are other ensembles; here similar considerations apply if α is designed to keep for instance the total energy constant*

Tests of the conjecture done in **2D***, [1], and weaker forms have been considered to accomodate* **3D** *simulations, [2], (which otherwise could be interpretable as negative results). The matter is at the moment open.*

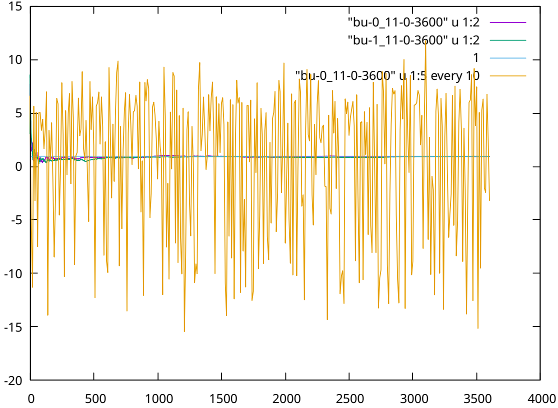
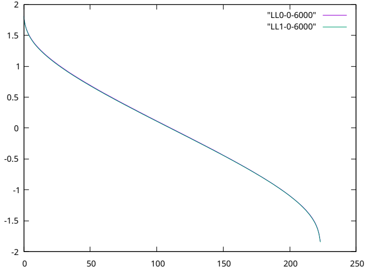


Figure: **Example** 15x15, 960 **armoniche,** $\nu = 2048^{-1}$**, 1-mode forcing**

Question: equivalence for nonlocal observ.? (as in Stat. Mech.)

Interesting non local observables are the Lyapunov exp. and the first of a series and it can be checked in a short time. In 2-dim. experiments : comparison between the Lyapunov spectra



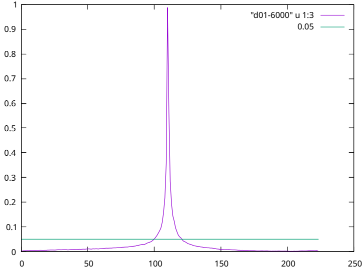


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Fig.2: Lyap.-spectra NS-RNS: This particular case: 224 d.o.f, viscosity 2^{-11}, time step $h = 2^{-17}$, forcing on a single $\pm$k.

It gives spectra close within 4% (away of $\lambda_i \simeq 0$, of course).
Checks equivalence of spectra NS-RNS:
$\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{\mathcal{N}-1}$, **in dDim** $\mathcal{N} = 2^{d-2}((2N+1)^d - 1)$.
Many new features emerge.

Remarkable pairing symmetry appears with N small, [3]

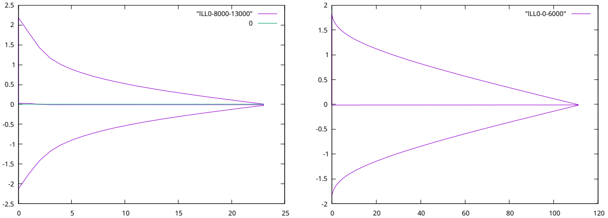

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Fig.3: Lyap.-spectra NS-RNS: Pairing of Lyap. exp.: 48 $(7x7)$ **and** 224 $(15x15)$ **d.o.f:** $\lambda_j + \lambda_{\mathcal{N}-1-j} = const$ **and const** $\simeq 0$. **Q.: Contradicts viscous dissip. ?**

The const=0 **is BUT suspect: should depend on UV cut-off** N**. Clearly if** N **is increased negative friction on small scales will prevail and dimension of attracting surface will go down and with it flow reversibility, Fluct. Th., pairing to** $0, \ldots$**.**

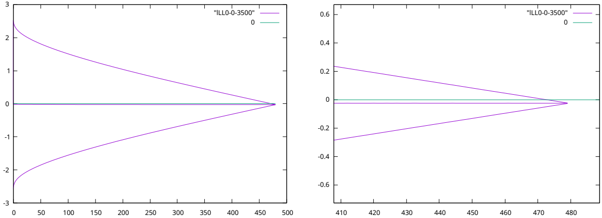

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Fig.4: Pairing of Lyap. exp.: 960 **d.o.f (**$15x15$**). The pairing remains but at level ¡0; KY dimension** $d = 900 < 960$**, [4].**

Tempting to interpret that attracting surface $\mathcal{A}$ **is** 900**-dim. However why pairing at** < 0 **value?**

The explanation I propose is that the flow on $\mathcal{A}$ remains reversible (with pairing at 0), but increasing further N the complete Lyapunov spectrum will give pairing, not to a constant but to a curve concave like a parabola. Reversibility on the attracting surface has been proposed and called "Axiom C", [5]. Would strongly support equivalence NS-RNS.

However it would not be time reversal symmetry (TRS) of the equations of motion ! which tranforms the attracting surface into a (disjoint) repelling one. The TRS would be spontaneously broken and replaced by a new symmetry whose existence is provided by the axiom C, if accepted, [6].

Of course the future will need test in simulations of all the properties implied by the CH, axiom C: an example beyond the cases mentioned above it that the FT might hold but with a different slope (in the 960 d.o.f. case above would be $\frac{900}{960}$).

Fluctuations are around average Lyap. exp. in NS and RNS.

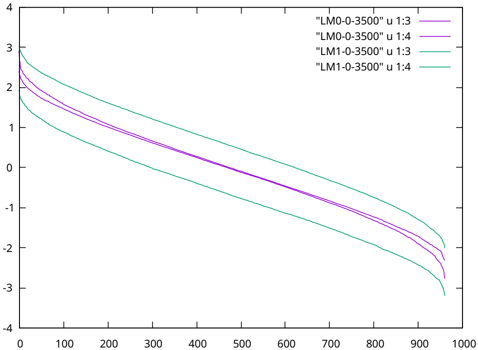

Figure: **The two inner lines give of each $k = 0, \ldots, 959$ the maximum and minimum of λ_k computed in 3500 iterations in the NS flow. The two external lines give the same for the RNS flow: the averages agree as in the previous drawing, but markedly different fluctuations**

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The subjects discussed here are being tested by several groups: here in Roma in particular

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