

Continuous quantum monitoring of position of nonlinear oscillators

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Application of the path-integral approach to continuous measurements leads to effective Lagrangians or Hamiltonians in which the effect of the measurement is taken into account through an imaginary term. We apply these considerations to nonlinear oscillators with use of numerical computations to evaluate quantum limitations for monitoring position in such a class of systems.

Several numerical experiments have shown that quantum-classical correspondence in chaotic dynamics seems to be anomalous [1]. In particular, it has been suggested that nonlinear systems showing chaotic behaviour in a classical regime may exhibit suppression of their stochasticity when quantization is taken into account [2]. However this suppression should appear on a time scale so short that the classical chaotic behaviour could not be observed in real systems. To recover the Liouville dynamics use of the quantum measurement process has been suggested [3]. In ref. [4] an example of a chaotic system in a quantum regime under instantaneous measurements has been investigated.

The quantum measurement processes for continuous monitoring of some observables of a dynamical system may be taken into account through the path-integral formalism [5–7] and offers the possibility to investigate in a general way suppression of quantum behaviour. It is therefore interesting to apply this method to nonlinear systems, specifically those pos-

sessing chaotic properties in the classical regime, having as a main goal the study of quantum suppression of chaotic behaviour in such a class of systems. In this Letter we deal with the development of a technique based upon numerical integration of an effective Schrödinger equation and applicable to continuous quantum measurements in nonlinear systems. As an example a nonlinear oscillator will be investigated.

The path-integral approach to continuous measurements is essentially based upon restriction of Feynman path integrals [5]. The integration is restricted to the set of paths compatible with the information, namely the given output of the continuous measurement. If the system moves between the points of the space-time $(x', 0)$ and (x'', τ) the result of integration $K_{\{a\}}(x'', \tau; x', 0)$ depends on the measurement output $[a] = \{a(t) | 0 \leq t \leq \tau\}$. This can be interpreted in two different ways. Firstly, it represents the probability amplitude for the measurement output $a(t)$ in the time interval τ given the po-

sitions of the system x' and x'' before and after the measurement. Secondly, it can be seen as a propagator for the system subject to the continuous measurement given the output of the measurement. The restriction of the path integral on some set of paths compatible with the measurement output can be effectively done with the help of a weight functional $w_{[a]}[x]$ depending on the measurement output $a(t)$ and decaying outside the set of paths compatible with $a(t)$,

$$K_{[a]}(x'', \tau; x', 0) = \int d[x] \exp\left(\frac{i}{\hbar} \int_0^\tau L(x, \dot{x}, t) dt\right) w_{[a]}[x]. \quad (1)$$

For example, if the coordinate x is monitored with error Δa and the result of the measurement is $a(t)$, the weight functional $w_{[a]}[x]$ selects those paths $x(t)$ in the corridor centered around $a(t)$ and having a width Δa (see fig. 1). A simple choice is a Gaussian weight functional

$$w_{[a]}[x] = \exp\left(-\frac{1}{\tau \Delta a^2} \int_0^\tau [x(t) - a(t)]^2 dt\right). \quad (2)$$

The restricted path integral is then

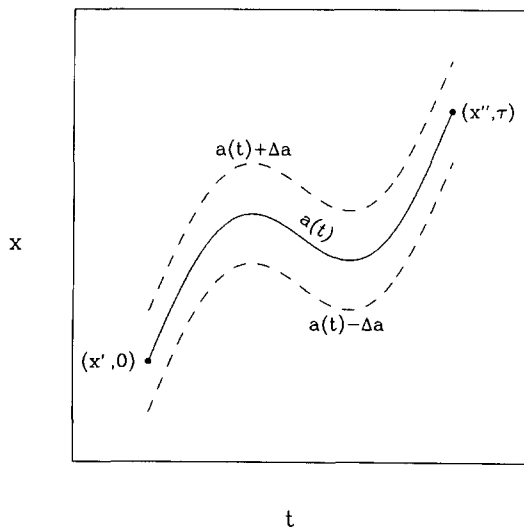


Fig. 1. A continuous measurement of position leads to a corridor centered at the measurement result $a(t)$ where the most probable paths lie.

$$K_{[a]}(x'', \tau; x', 0) = \int d[x] \exp\left(\frac{i}{\hbar} \int_0^\tau L(x, \dot{x}, t) dt - \frac{1}{\tau \Delta a^2} \int_0^\tau (x - a)^2 dt\right). \quad (3)$$

It is important to observe that the resulting path integral may be considered as describing a free (i.e. not measured) system but with effective Lagrangian having an imaginary term due to the measurement [8],

$$L_{\text{eff}}(x, \dot{x}, t) = L(x, \dot{x}, t) + \frac{i\hbar}{\tau \Delta a^2} [x - a(t)]^2. \quad (4)$$

This contribution produces a decrease of the density of the system in the configuration space far from $x(t) = a(t)$. The decrease is linked with the restriction of the alternatives due to the measurement performed^{#1}. Finally the effective Lagrangian is time-dependent even if the original Lagrangian is not.

In order to estimate a probability distribution for the measurement output of a physical system we consider the convolution

$$I_{[a]} = \langle \phi_2 | K_{[a]} | \phi_1 \rangle = \iint \phi_2^*(x'') K_{[a]}(x'', \tau; x', 0) \phi_1(x') dx' dx'' \quad (5)$$

According to the first interpretation of $K_{[a]}$ the quantity $I_{[a]}$ is a probability amplitude for the measurement to give the output $a(t)$ under the condition that the system has been in the state ϕ_1 before the measurement and in the state ϕ_2 after a time τ . The probability distribution for the measurement output is then

$$P_{[a]} = \frac{|I_{[a]}|^2}{\int d[a] |I_{[a]}|^2}. \quad (6)$$

Note that $P_{[a]}$ depends upon the instrumental uncertainty Δa .

The amplitude $I_{[a]}$ can be also written as a scalar product,

$$I_{[a]} = \langle \phi_2 | \psi_{[a]}(\tau) \rangle, \quad (7)$$

^{#1} For a detailed discussion of this effect and the related irreversibility introduced during the measurement see ref. [9].

where

$$\psi_{[a]}(x'', \tau) = \int K_{[a]}(x'', \tau; x', 0) \phi_1(x') dx'. \quad (8)$$

According to the second interpretation of $K_{[a]}$ the wave function $\psi_{[a]}(x, t)$ represents the evolution at time t of the state $\phi_1(x)$ under the action of a continuous measurement with output $a(t)$. This also means that $\psi_{[a]}(x, t)$ can be found as the solution of the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi_{[a]}(x, t)}{\partial t} = H_{\text{eff}} \psi_{[a]}(x, t) \quad (9)$$

with an effective Hamiltonian H_{eff} corresponding to the Lagrangian L_{eff} in (4) and with the choice $\psi_{[a]}(x, 0) = \phi_1(x)$.

The time dependence of the effective Hamiltonian H_{eff} makes analytical calculations difficult. In this case one may more simply evaluate $P_{[a]}$ through the Feynman propagator $K_{[a]}$. However, due to the quadratic nature of the measurement contribution to L_{eff} , analytical calculations are essentially restricted to the case in which $L(x, \dot{x}, t)$ is a linear oscillator. In general a numerical approach must be followed and in this case the Schrödinger formalism is more suitable. The partial differential equation (9) is reduced to a simple finite difference recursive equation by choosing a proper lattice to simulate the continuous space-time [10]. No particular problems arise from the time dependence and the non-Hermitian nature of the differential operator H_{eff} [11].

In order to understand the behaviour of a nonlinear system let us first consider a linear oscillator

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2. \quad (10)$$

In this case analytical results have been obtained allowing a test of the numerical technique. The effective Lagrangian corresponds to a forced linear oscillator

$$L_{\text{eff}} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega_r^2 x^2 - \frac{2i\hbar}{\tau \Delta a^2} a(t) x + \frac{i\hbar}{\tau \Delta a^2} a(t)^2 \quad (11)$$

with renormalized complex frequency

$$\omega_r^2 = \omega^2 - \frac{2i\hbar}{m\tau \Delta a^2}. \quad (12)$$

For any choice of the measurement output $a(t)$ the

propagating kernel $K_{[a]}$ can be easily calculated [12]. Let us consider what happens when the quantum system is in the ground state of the unmeasured oscillator before and after the period τ of the continuous measurement,

$$\phi_1(x) = \phi_2(x) = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar} x^2 \right). \quad (13)$$

Due to the shape of ϕ_1 and ϕ_2 it is natural to choose the null boundary conditions $a(0) = a(\tau) = 0$ for the measurement output $a(t)$. Any such function $a(t)$ can be written as a Fourier sine series. We consider only measurement outputs of the form

$$a(t) = \epsilon \sin \Omega_n t, \quad \Omega_n = n\pi/\tau, \quad (14)$$

where n is an integer number. For a fixed Ω_n the probability distribution $P_{[a]}$ is reduced to a function of the amplitude ϵ . $P(\epsilon)$ is a Gaussian function of width Δa_{eff} ,

$$P(\epsilon) = \frac{1}{\sqrt{\pi} \Delta a_{\text{eff}}} \exp\left(-\frac{\epsilon^2}{\Delta a_{\text{eff}}^2} \right), \quad (15)$$

where

$$\Delta a_{\text{eff}}^{-2} = 2 \operatorname{Re} \left\{ \frac{1}{2\Delta a^2} \left(1 - \frac{2i\hbar}{m\tau \Delta a^2 (\Omega_n^2 - \omega_r^2)} - \frac{4\hbar \Omega_n^2}{m\omega \tau^2 \Delta a^4 (\Omega_n^2 - \omega_r^2)} \times \left[1 - i \frac{\omega_r}{\omega} \left(\cot(\omega_r \tau) + \frac{(-1)^n}{\sin(\omega_r \tau)} \right) \right]^{-1} \right) \right\}. \quad (16)$$

The meaning of Δa_{eff} is linked to the role of the measurement device during the evolution of the system under measurement. When the instrument error Δa is large in comparison to the characteristic quantum scale of the system under measurement, $\sqrt{\hbar/m\omega}$ in this case, the corridor shown in fig. 1 contains the classical trajectory (with null boundary conditions). Thus the classical limit is

$$\Delta a_c \equiv \lim_{\Delta a \rightarrow \infty} \Delta a_{\text{eff}} = \Delta a. \quad (17)$$

On the other hand when Δa becomes small quantum noise arises. Also corridors far from the classical trajectory are probable. The quantum limit is written as

$$\Delta a_q \equiv \lim_{\Delta a \rightarrow 0} \Delta a_{\text{eff}} = \left[\left(\frac{m}{\hbar} \right)^{3/2} \tau^{1/2} \Omega_n^2 \Delta a + \left(\frac{m\tau}{2\hbar} \right)^2 (\Omega_n^2 - \omega^2)^2 \Delta a^2 \right]^{-1/2}, \quad (18)$$

i.e. the effective error Δa_{eff} diverges as $\Delta a^{-1/2}$. In an intermediate situation Δa_{eff} interpolates between these two limits always maintaining values larger than the instrumental error. In fig. 2 the behaviour of Δa_{eff} versus Δa is shown for three different values of the ratio Ω_n/ω . As shown in (17) and (18), while the classical limit is the same for all the situations, different behaviours appear in the quantum regime. The minimum effective uncertainty is maximum at the resonance condition $\Omega_n = \omega$. In this case always $\Delta a_{\text{eff}} \geq \sqrt{\hbar/m\omega}$. When $\Omega_n \neq \omega$ the minimum effective uncertainty estimated by the intersection between quantum and classical limits decreases as $(\Omega_n^2 - \omega^2)^{-1}$.

The comparison between analytical and numerical results for Δa_{eff} are shown in fig. 3 for two different

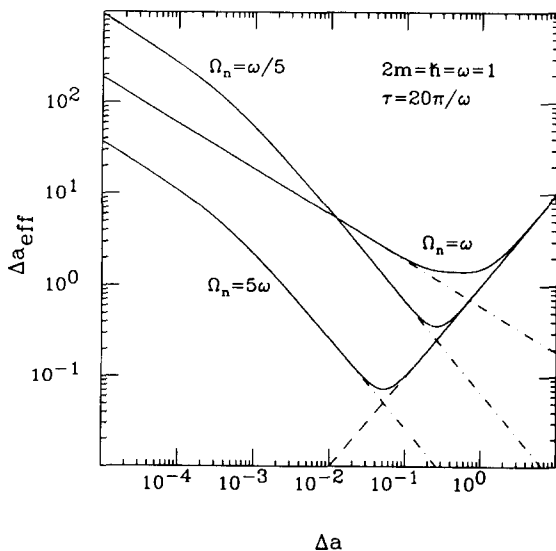


Fig. 2. Effective uncertainty versus instrumental uncertainty for three different values of the ratio Ω_n/ω in the case of a linear oscillator. The dashed line is the classical limit Δa_c , the dot-dashed lines are the quantum limits Δa_q . The value of the other parameters is also indicated.

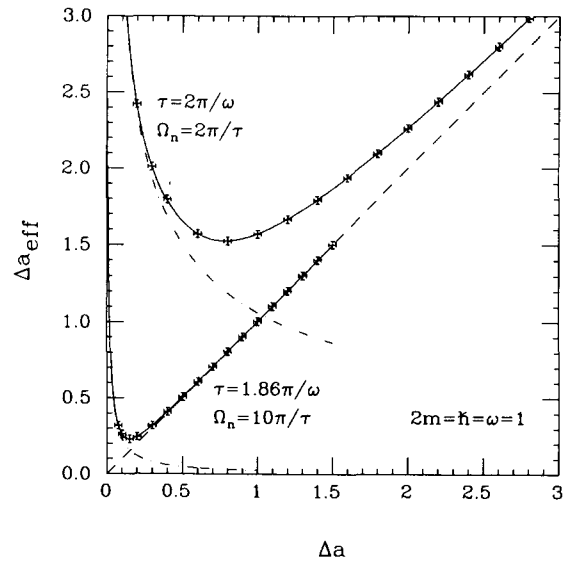


Fig. 3. Numerical (dots) and analytical (solid line) effective uncertainty versus instrumental uncertainty for a linear oscillator. Asymptotic classical (dashed) and quantum (dot-dashed) behaviours are also shown.

choices of the measurement parameters. Also shown are classical and quantum behaviours, corresponding to the limits expressed in (17) and (18). The difference between numerical and analytical results is less than 0.1% and it can be further reduced by choosing higher resolution space-time lattices.

The numerical accuracy estimated above allows us to perform meaningful computations for a nonlinear oscillator represented by the Lagrangian

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 - \frac{1}{4} \beta x^4. \quad (19)$$

For comparison with the previous linear case we have chosen the initial and final states of the form (13) and a measurement output of the form (14). In this case we expect both non-Gaussian behaviours for the wave function at $t = \tau$ and the propagator. This also means that the distribution $P(\epsilon)$ is not a Gaussian function. An equivalent width for $P(\epsilon)$ may be introduced through the definition

$$\Delta a_{\text{eff}} \equiv \frac{1}{\sqrt{\pi} P(0)} \int_{-\infty}^{\infty} P(\epsilon) d\epsilon. \quad (20)$$

The computed Δa_{eff} versus Δa are shown in fig. 4 for two different values of the nonlinearity coefficient β .

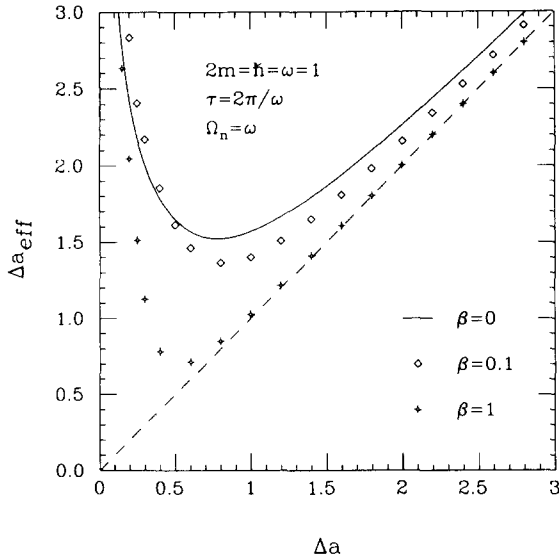


Fig. 4. Numerical (dots) effective uncertainty versus instrumental uncertainty for the nonlinear oscillator at two different values of β . The solid line is the behaviour of the corresponding linear oscillator obtained for $\beta=0$. The dashed line represents the classical limit.

The comparison with the linear situation having the same parameters is also shown. It appears that the effect of the nonlinear term is in the direction to enlarge the region in which the classical approximation is meaningful. The quartic term concentrates the final wave function near the result of the measurement and this implies that the effect of quantum noise, toward a spreading of the most probable paths, is reduced.

In all the previous considerations we have chosen to deal with Δa which is time independent. A more general class of continuous measurements is obtained by considering time dependence for Δa . In particular to recover chaotic dynamics through quantum measurements particular conditions on the kind of measurements must be satisfied, namely the

measurement process has to be a quantum nondemolition process for the observable under monitoring [4]. Quantum nondemolition strategies for linear oscillators have been already analyzed in the framework of the path integral with continuous measurements [8]. The application of the technique described here for quantum nondemolition measurement processes on nonlinear systems will be the subject of future investigations.

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