Measurement Quantum Mechanics and Experiments on Quantum Zeno Effect

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Measurement quantum mechanics, the theory of a quantum system which undergoes a measurement process, is introduced by a loop of mathematical equivalencies connecting previously proposed approaches. The unique phenomenological parameter of the theory is linked to the physical properties of an informational environment acting as a measurement apparatus which allows for an objective role of the observer. Comparison with a recently reported experiment suggests how to investigate novel interesting regimes for the quantum Zeno effect. © 1996 Academic Press, Inc.

1. INTRODUCTION

The description of the measurement process has been a topic debated from the early developments of quantum mechanics [1–3]. Besides being a basic issue in the interpretation of the quantum formalism it has also a practical interest in predicting the results of experiments pointing out some paradoxical aspects of the quantum laws when compared to the classical ones. The development of devices whose noise figures are close to the quantum limit makes these experiments within the technological feasibility and demands for a systematization of the theory of quantum measurement with deeper understanding of its experimental implications [4, 5].

In ordinary quantum mechanics measurements are taken into account by postulating the wave function collapse [1]. This approach has the unpleasant

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feature of introducing an extra assumption in the theory which, moreover, regards only instantaneous and perfect measurements. In the last twenty years relevant steps have been made in upgrading the von Neumann postulate with more realistic and satisfactory approaches. These approaches essentially recognize that a measured system is not isolated but in interaction with a measurement apparatus. The way this basic fact is taken into account in the evolution law of the measured system has been developed according to different languages and points of view.

A group algebra approach to the problem of an open quantum system was proposed by postulating (completely positive) semigroup properties for the dynamical law of a system in interaction with a Markovian environment [6, 7]. In this case the open system is described by a semigroup master equation later successfully used in modeling quantum optics experiments [8]. The semigroup master equation, still preserving positivity and trace of the density matrix operator, introduces decoherentization by dynamically quenching the off-diagonal elements of the density matrix, a key property used to explain the absence of superposition states in a measurement apparatus [9] or in a general macroscopic system [10, 12] thought as systems interacting with an environment. Open quantum systems were gathered to measured systems by obtaining an all alike semigroup master equation for a continuous measurement process [12] modeled by repeated instantaneous effect-valued measurements [13], i.e., partial localization or decoherentization kicks [14] given to the density matrix at random values of the measured observable.

In the model [12] the semigroup master equation comes out only after averaging the instantaneous results of the measured observable with probability distribution in agreement with standard quantum mechanics [15]. In absence of this average, i.e., for a particular *selection* of the measurement outcomes, a nonlinear stochastic differential equation was proposed to describe the evolution of the density matrix during a continuous measurement process [16]. Since in this case during the measurement process the density matrix coincides with its square, a nonlinear stochastic differential equation could be derived for the corresponding wave function [17] crowning previous attempts to include the Lüders postulate [18], a generalization of the von Neumann one to *selective* measurements, into a stochastic Schrödinger equation [19]. This equation, obtained also in the framework of the quantum filtering theory [20], is a special case of the more general quantum state diffusion equation for open systems [21]. Its nonlinearity can be avoided only renouncing to make predictions on the outcome of the measured observable. Indeed, a linear stochastic differential equation was proposed for the so called *a posteriori* states [22, 23], i.e., those unnormalized states which describe the quantum system when the measurement result is already known.

As originally suggested by Feynman [24] the path integral approach to quantum mechanics is a quite appropriate framework for incorporating the effect of a measurement. The hint was picked up by modeling a continuous measurement of position having a certain result (selective measurement) by a restriction of the Feynman path integral [25]. The method was extended to measurements of a generic observable, function of momentum and position, by a restriction of the

quantum propagator in the phase-space formulation [26, 27]. The restriction was originally proposed as a Gaussian functional damping the paths proportionally to the time-averaged squared difference between the value taken by the observable on the paths and the measurement result, normalized to a given variance [28]. However, the corresponding quantum propagator has desirable dynamical semi-group properties if the Gaussian restriction is chosen linear in time [29] which is equivalent to make the previously proposed variance scaling with the inverse of the measurement time [30].

The modifications introduced in the quantum propagator by the restricted path integral approach can be incorporated into an effective Hamiltonian depending on the selected measurement result. An effective wave equation was then derived which is linear in the wave function if the selected measurement result is considered known [31–33]. On the other hand, if the selected measurement result is to be determined according to the evolution of the measured system the effective wave equation is read as a nonlinear stochastic differential equation. Equivalence to the previously proposed stochastic equations occurs [34].

The selective constraint imposed to the restricted path integral approach can be removed by summing over all possible measurement outcomes. In this way a non-selective process described by a density matrix was obtained [35]. The effect of the interaction with the measurement apparatus modifies the corresponding quantum propagator through an influence functional [36] which turns out to be equivalent to those obtained with other methods [12, 37, 38].

From the above incomplete list of approaches, apart from an evident difference in the languages, a rather unified picture of the problem of quantum measurement emerges. It deserves uprising to the systematic theory of a quantum system evolving under the effect of a measurement process. Such theory, named, for brevity, measurement quantum mechanics, is formally presented in Section II.

A less formal introduction to measurement quantum mechanics is attainable through the analysis of a model of measurement device which is rather general. This is important not only as a justification of the formal approach to the theory but, above all, for clearly defining the meaning of a measurement in relation to the observer. As noted by Cini [39] paradoxical features arise from not considering the objective role of the observer in a measurement process. The evolution of a measured system in interaction with a measurement apparatus can not depend on the observer looking or not at the pointer of the apparatus. In order to take into account such objectivity the measurement apparatus must be classical with respect to the observer [39]. In Section III we propose a simple model of measurement apparatus shaped as an environment of particles linearly interacting with the measured system and in contact with a heat reservoir at a fixed temperature [40]. In the high temperature limit the particles behave as an informational environment which extracts information in objective way. Noticeably, the formal structure of measurement quantum mechanics is obtained in the same limit.

A further analysis of measurement quantum mechanics is given in Section IV in connection to a recent optical experiment [41] showing quantum Zeno effect.

Comparison of the experimental results with the theoretical predictions indicate that the experiment [41] was performed in a regime of very strong coupling of the measurement apparatus with the measured system. Repetition of this experiment or similar ones in a weaker coupling regime is desirable for investigating interesting quantum features.

Some final remarks are given in Section V.

II. MEASUREMENT QUANTUM MECHANICS

In this section we show the mathematical equivalence of the five approaches to the problem of measurement in quantum systems briefly described in the introduction. The demonstration is organized in steps relating neighboring pairs of approaches and giving rise to the equivalence loop sketched in Fig. 1. No one of these equivalence steps is novel. However, we reconsider all them in a unified framework and language with naturally emerging definitions for concepts as selective and nonselective measurements as well as *a priori* and *a posteriori* analysis of a measurement process. As a result we get a theory which takes into account the effect of a measurement process in ordinary quantum dynamics through a phenomenological parameter coupling the measured system to the measurement
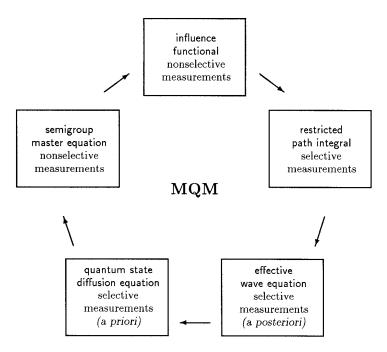

FIG. 1.   Measurement quantum mechanics: connections among different approaches.

apparatus. We enter the loop of Fig. 1 at the group algebra approach to the master equation and go on in the clockwise sense.

The dynamics of a system interacting with an environment is conveniently described in terms of a reduced density matrix operator $\hat{\rho}(t)$ obtained by tracing out the environment variables from the density matrix operator of the whole system + environment. The unitary evolution of $\hat{\rho}(t)$ for the isolated system is modified to an irreversible one by the interaction with the environment. In the limit of a Markovian environment a dynamical law described by a completely positive semigroup has been postulated [6, 7] resulting in the following master equation for the reduced density matrix operator

$$\frac{d}{dt}\hat{\rho}(t) = -\frac{i}{h}[\hat{H}(t), \hat{\rho}(t)] + \frac{1}{2}\sum_{\nu}([\hat{L}_{\nu}(t)\,\hat{\rho}(t),\,\hat{L}_{\nu}(t)^{\dagger}] + [\hat{L}_{\nu}(t),\,\hat{\rho}(t)\,\hat{L}_{\nu}(t)^{\dagger}]) \quad (1)$$

where $\hat{H}(t) = \hat{H}(\hat{p}, \hat{q}, t)$ is the Hamiltonian operator for a general nonautonomous system and $\hat{L}_{\nu}(t)$ are the Lindblad operators representing the influence of the environment on the system.

The above equation is thought to describe the general case of an open quantum system. The evolution of a quantum system subjected to a measurement process is a particular case where the environment is the measurement apparatus and the Lindblad operators are proportional to the measured quantities. If, for simplicity, we consider the measurement of a single observable represented by the Hermitian operator $\hat{A}(t) = \hat{A}(\hat{p}, \hat{q}, t)$, the corresponding Lindblad operator can be chosen as $\hat{L}(t) = \hat{L}(t)^{\dagger} = \kappa(t)^{1/2}\,\hat{A}(t)$. Note that we allow for an explicit time dependence of $\hat{A}(t)$ in order to include the case of general observables such as the continuous quantum nondemolition ones [42]. The function $\kappa(t)$ has dimensions $[\kappa] = [t^{-1}A^{-2}]$ and represents, as we shall see in the next section, the coupling of the measured system to the measurement apparatus. The measurement process is continuous in time and the measurement coupling is time dependent as requested in the description of a general experimental situation.

Due to the presence of commutators in the right hand side of (1) the trace of the reduced density matrix operator is a conserved quantity which we assume to be unity

$$\text{Tr } \hat{\rho}(t) = 1. \quad (2)$$

We shall see later, when introducing selective measurements, that the reduced density matrix operator $\hat{\rho}(t)$ corresponds to an incoherent mixture of pure states associated to selective processes. In antithesis, the process described by $\hat{\rho}(t)$ is called nonselective and Eq. (2) can be interpreted as a normalization relation for the probability distribution over the selective processes. According to this interpretation, the result of a nonselective measurement $\overline{a(t)}$ and its associated variance $\Delta a(t)^2$, can be evaluated by the trace rules

$$\overline{a(t)} = \text{Tr}[\hat{A}(t)\,\hat{\rho}(t)] \quad (3)$$

and

$$\Delta a(t)^2 = \mathrm{Tr}[\,(\hat{A}(t) - \overline{a(t)})^2 \,\hat{\rho}(t)\,], \tag{4}$$

respectively. Here overlining is used to denote the statistical average over the selective measurement results $a(t)$ to be defined in the following together with the corresponding probability distribution.

The equivalence of the master Equation (1) to a density matrix propagator expressed in terms of an influence functional is our first step in the loop of Fig. 1. By moving to the coordinate representation of the reduced density matrix operator

$$\rho(q_1, q_2, t) = \langle q_1 | \,\hat{\rho}(t)\, | q_2 \rangle \tag{5}$$

the corresponding evolution equation becomes the partial differential equation

$$\frac{\partial}{\partial t} \rho(q_1, q_2, t) = \left[-\frac{i}{h} H\left(-ih \frac{\partial}{\partial q_1}, q_1, t \right) + \frac{i}{h} H\left(-ih \frac{\partial}{\partial q_2}, q_2, t \right) \right.$$

$$\left. -\frac{1}{2} \kappa(t) \left[A\left(-ih \frac{\partial}{\partial q_1}, q_1, t \right) - A\left(-ih \frac{\partial}{\partial q_2}, q_2, t \right) \right]^2 \right] \rho(q_1, q_2, t) \tag{6}$$

which can be transformed into the integral equation

$$\rho(q_1'', q_2'', t'') = \int dq_1' \, dq_2' \, G(q_1'', q_2'', t''; q_1', q_2', t') \, \rho(q_1', q_2', t'). \tag{7}$$

The two-point Green function (density matrix propagator) G has a phase-space path-integral representation which can be derived by standard methods [35, 40]. For a small time interval Δt one has

$$\langle q_1'' | \,\hat{\rho}(t'')\, | q_2'' \rangle = \langle q_1'' | \,\hat{\rho}(t'' - \Delta t)\, | q_2'' \rangle + \langle q_1'' | \,\frac{d}{dt} \,\hat{\rho}(t'' - \Delta t)\, | q_2'' \rangle \, \Delta t + \mathcal{O}(\Delta t^2). \tag{8}$$

By using Eq. (1) and inserting at the appropriate places the four identities

$$\int dp_1^{(1)} \, | p_1^{(1)} \rangle \langle p_1^{(1)} | \int dq_1^{(1)} \, | q_1^{(1)} \rangle \langle q_1^{(1)} | \int dp_2^{(1)} \, | p_2^{(1)} \rangle \langle p_2^{(1)} | \int dq_2^{(1)} \, | q_2^{(1)} \rangle \langle q_2^{(1)} | \tag{9}$$

where

$$\langle q | p \rangle = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{i}{h} pq \right) \tag{10}$$

we get

$$
\rho(q_1'', q_2'', t'') = \int \frac{dp_1^{(1)}}{2\pi h} \, dq_1^{(1)} \int \frac{dp_2^{(1)}}{2\pi h} \, dq_2^{(1)}
$$

$$
\times \exp\left(\frac{i}{h} \left[p_1^{(1)} \frac{q_1'' - q_1^{(1)}}{\Delta t} - H(p_1^{(1)}, q_1^{(1)}, t'' - \Delta t) \right] \Delta t \right.
$$

$$
- \frac{1}{2} \kappa(t'' - \Delta t) \left[A(p_1^{(1)}, q_1^{(1)}, t'' - \Delta t) - A(p_2^{(1)}, q_2^{(1)}, t'' - \Delta t) \right]^2 \Delta t
$$

$$
\left. - \frac{i}{h} \left[p_2^{(1)} \frac{q_2'' - q_2^{(1)}}{\Delta t} - H(p_2^{(1)}, q_2^{(1)}, t'' - \Delta t) \right] \Delta t \right)
$$

$$
\times \rho(q_1^{(1)}, q_2^{(1)}, t'' - \Delta t) + \mathcal{O}(\Delta t^2). \tag{11}
$$

By iterating this relation N times with $\Delta t = (t'' - t')/N$ and then taking the limit $N \to \infty$, a functional measure arises

$$
d[p] \, d[q]_{q', t'}^{q'', t''} = \lim_{N \to \infty} \prod_{n=1}^{N} \frac{dp^{(n)}}{2\pi h} \prod_{n=1}^{N-1} dq^{(n)} \tag{12}
$$

with boundary conditions imposed only to the $q(t)$ paths. By comparison with Eq. (7) we conclude

$$
G(q_1'', q_2'', t''; q_1', q_2', t') = \int d[p_1] \, d[q_1]_{q_1', t'}^{q_1'', t''} \int d[p_2] \, d[q_2]_{q_2', t'}^{q_2'', t''}
$$

$$
\times \exp\left(\frac{i}{h} S[p_1, q_1] - \frac{i}{h} S[p_2, q_2] - Z[p_1, q_1, p_2, q_2] \right) \tag{13}
$$

where

$$
S[p, q] = \int_{t'}^{t''} dt [p\dot{q} - H(p, q, t)] \tag{14}
$$

$$
Z[p_1, q_1, p_2, q_2] = \tfrac{1}{2} \int_{t'}^{t''} dt \, \kappa(t) [A(p_1, q_1, t) - A(p_2, q_2, t)]^2. \tag{15}
$$

The effect of the measurement in the two-point Green function is represented by the functional $\exp(-Z)$ which reduces to the identity for $\kappa(t) = 0$, i.e., in absence of a measurement process. The functional $\exp(-Z)$ is the Feynman–Vernon influence functional [36] evaluated by tracing out the degrees of freedom of the environment in the phase-space path-integral formulation of quantum mechanics. As we shall see in the next section, the influence functional approach allows us to give an

enlightening interpretation of the nonselective measurement processes with an explicit expression of the parameter $\kappa(t)$.

The second step in the equivalence loop of Fig. 1 is accomplished by a formal manipulation of the influence functional [35]. By using the identity

$$\exp\left(-\frac{1}{2}\int_{t'}^{t''} dt\,\kappa(t)[A(p_1, q_1, t) - A(p_2, q_2, t)]^2\right)$$

$$= \int d[a] \exp\left(-\int_{t'}^{t''} dt\,\kappa(t)[A(p_1, q_1, t) - a(t)]^2 - \int_{t'}^{t''} dt\,\kappa(t)[A(p_2, q_2, t) - a(t)]^2\right) \tag{16}$$

where the functional measure arises by slicing the interval $[t', t'']$ into $N \to \infty$ sub-intervals at times $t^{(n)} = t'' - n\,\Delta t$ with $\Delta t = (t'' - t')/N$, i.e.,

$$d[a] = \lim_{N \to \infty} \prod_{n=1}^{N} da^{(n)} \sqrt{\frac{2\kappa(t^{(n)})\,\Delta t}{\pi}}, \tag{17}$$

the two-point Green function G can be decomposed into a couple of one-point Green functions $G_{[a]}$

$$G(q_1'', q_2'', t''; q_1', q_2', t') = \int d[a]\, G_{[a]}(q_1'', t''; q_1', t')\, G_{[a]}(q_2'', t''; q_2', t')^* \tag{18}$$

where

$$G_{[a]}(q'', t''; q', t') = \int d[p]\, d[q]_{q', t'}^{q'', t''} \exp\left(\frac{i}{\hbar} S[p, q] - \int_{t'}^{t''} dt\,\kappa(t)[A(p, q, t) - a(t)]^2\right). \tag{19}$$

This one-point Green function is the phase-space generalization of the restricted path-integral approach originally proposed on physical grounds as a damping of the Feynman paths incompatible with the measurement result [25]. Here the measurement result is the function $a(t)$ which, according to the interpretation of the functional measure $d[a]$, is, in general, continuous but not differentiable. If we assume that the measured system is in a pure state $|\psi(t')\rangle$ at the beginning of the measurement process, i.e.,

$$\rho(q_1', q_2', t') = \langle q_1' | \psi(t')\rangle\langle\psi(t') | q_2'\rangle = \psi(q_1', t')\,\psi(q_2', t')^*, \tag{20}$$

Eq. (18) allows us to introduce a pure state $|\psi_{[a]}(t)\rangle$ which is the evolution at time $t > t'$ of the initial one $|\psi(t')\rangle$. Indeed, the reduced density matrix can be thought as the functional integral

$$\rho(q_1, q_2, t) = \int d[a]\, \psi_{[a]}(q_1, t)\, \psi_{[a]}(q_2, t)^* \tag{21}$$

with the wave function of the pure state defined by the one-point Green function $G_{[a]}$

$$\psi_{[a]}(q'', t'') = \int dq' \, G_{[a]}(q'', t''; q', t') \, \psi(q', t'). \qquad (22)$$

The decomposition (18) introduces the announced statistical description of a non-selective measurement process in terms of selective processes corresponding to different measurement results $a(t)$ and characterized by a certain probability distribution. Due to Eq. (21) the conservation of the trace of the reduced density matrix operator gives

$$1 = \int d[a] \| \psi_{[a]}(t) \|^2. \qquad (23)$$

Therefore, $\| \psi_{[a]}(t) \|^2$ is the probability distribution for the selective process with measurement result $[a]$. Once the probability distribution is known, the nonselective measurement result (3) and variance (4) which, according to (21), are written also as

$$\overline{a(t)} = \int d[a] \langle \psi_{[a]}(t) | \, \hat{A}(t) \, | \psi_{[a]}(t) \rangle \qquad (24)$$

and

$$\Delta a(t)^2 = \int d[a] \, \langle \psi_{[a]}(t) | \, (\hat{A}(t) - \overline{a(t)})^2 \, | \psi_{[a]}(t) \rangle, \qquad (25)$$

can be expressed in terms of the selective results $a(t)$ by

$$\overline{a(t)} = \int d[a] \| \psi_{[a]}(t) \|^2 \, a(t) \qquad (26)$$

and

$$\Delta a(t)^2 = \lim_{\tau \to 0^+} \int d[a] \| \psi_{[a]}(t) \|^2 \, (a(t) - \overline{a(t)})(a(t - \tau) - \overline{a(t)}), \qquad (27)$$

respectively. The equivalence between (24) and (26) as well as that one between (25) and (27) can be demonstrated explicitly by using the definition of the functional measure $d[a]$ and the functional expression of the wave function $\psi_{[a]}(q, t)$. The prescription $\tau \to 0^+$ in (27) is used to avoid the divergence which occurs by integrating the term $a(t)^2$.

The third step of our equivalence loop, namely the existence of a differential equation for the wave function $\psi_{[a]}(q, t)$, follows from the same standard method

used for the reduced density matrix. The explicit representation of the functional integral giving the one-point Green function $G_{[a]}$ allows us to write the difference of the wave function between two close times t and $t + \Delta t$ up to terms $\mathcal{O}(\Delta t^2)$ and find the differential equation

$$\frac{\partial}{\partial t} \psi_{[a]}(q, t) = \left[-\frac{i}{h} H\left(-ih \frac{\partial}{\partial q}, q, t \right) - \kappa(t) \left[A\left(-ih \frac{\partial}{\partial q}, q, t \right) - a(t) \right]^2 \right] \psi_{[a]}(q, t).$$

(28)

The mathematical properties of this differential equation crucially depend on two possible ways of analyzing a selective measurement process. A first possibility is to consider an analysis *a posteriori*. The result of the measurement $a(t)$ is already known and one wants to complete the quantum mechanical information on the measured system by evaluating the associated wave function $\psi_{[a]}(q, t)$. In this case Eq. (28) is an effective wave equation linear in the wave function. This equation is deterministic but not regular, in general, depending upon the nature of the function $a(t)$. The effect of the measurement appears as an anti-Hermitian term added to the Hamiltonian of the unmeasured system. Due to this term the norm of the wave function $\psi_{[a]}(q, t)$ is not conserved in agreement with the probabilistic normalization (23).

A second possibility is to consider an analysis *a priori*. The result of the measurement $a(t)$ is to be predicted not deterministically but randomly with probability distribution $\|\psi_{[a]}(t)\|^2$. Equation (28) becomes a stochastic differential equation nonlinear in $\psi_{[a]}(q, t)$. This can be seen more explicitly by introducing an appropriate noise $\eta_{[a]}(t)$ which relates the selective result $a(t)$ to the expectation value of the operator $\hat{A}(t)$ in the state $|\psi_{[a]}(t)\rangle$. Comparison of (24) and (26) allows us to write

$$a(t) = \frac{\langle \psi_{[a]}(t)| \hat{A}(t) |\psi_{[a]}(t)\rangle}{\langle \psi_{[a]}(t) | \psi_{[a]}(t)\rangle} + \eta_{[a]}(t)$$

(29)

where $\eta_{[a]}(t)$ is defined by

$$\int d[a] \, \|\psi_{[a]}(t)\|^2 \, \eta_{[a]}(t) = 0.$$

(30)

The nonlinearity of the stochastic equation for $\psi_{[a]}(q, t)$ is made clear by inserting (29) into (28). This stochastic equation is also unconventional in the sense that it contains the square of the noise term $\eta_{[a]}(t)$.

It is evident that Eq. (28) is fully appropriate in the *a posteriori* analysis of a selective measurement process. However, in the *a priori* analysis one has to deal with an unconventional stochastic equation containing a complicated noise function defined in terms of a time-dependent probability distribution. It would be more desirable to have a standard stochastic equation with a white noise. This can be

achieved, together with the fourth step in our equivalence loop, by a change of variable in the measurement result. Analogously to Eq. (29) we write

$$a(t) = a_{[\xi]}(t) + \frac{\xi(t)}{2\sqrt{\kappa(t)}}$$ (31)

where $a_{[\xi]}(t)$ is considered known for the moment and $\xi(t)$ is a noise whose characterization is obtained through the following considerations. The two-point Green function already decomposed in (18) into a couple of one-point Green functions $G_{[a]}$ with functional measure $d[a]$ can alternatively be decomposed into another couple of one-point Green functions $G_{[\xi]}$

$$G(q_1'', q_2'', t''; q_1', q_2', t') = \int d[\xi]\, G_{[\xi]}(q_1'', t''; q_1', t')\, G_{[\xi]}(q_2'', t''; q_2', t')^*$$ (32)

with Gaussian measure

$$d[\xi] = \lim_{N \to \infty} \prod_{n=1}^{N} d\xi^{(n)} \sqrt{\frac{\Delta t}{2\pi}} \exp\left(-\tfrac{1}{2}\, \xi^{(n)2} \Delta t\right)$$ (33)

which makes $G_{[\xi]}$ exponentially linear in ξ

$$G_{[\xi]}(q'', t''; q', t') = \exp\left(\int_{t'}^{t''} dt\, \frac{1}{4}\, \xi(t)^2\right) G_{[a]}(q'', t''; q', t')$$

$$= \int d[p]\, d[q]_{q',t'}^{q'',t''} \exp\left(\frac{i}{\hbar} S[p,q] - \int_{t'}^{t''} dt\, \kappa(t)\right.$$

$$\left. \times \left[[A(p,q,t) - a_{[\xi]}(t)]^2 - 2[A(p,q,t) - a_{[\xi]}(t)] \frac{\xi(t)}{2\sqrt{\kappa(t)}} \right] \right).$$ (34)

With respect to the Gaussian measure $d[\xi]$, the noise $\xi(t)$ has average

$$\overline{\xi(t)} = \int d[\xi]\, \xi(t) = 0$$ (35)

and covariance

$$\overline{\xi(t_1)\, \xi(t_2)} = \int d[\xi]\, \xi(t_1)\, \xi(t_2) = \delta(t_1 - t_2),$$ (36)

i.e., it is a white noise. As in the previous case, the reduced density matrix can be written as a functional integral

$$\rho(q_1, q_2, t) = \int d[\xi]\, \psi_{[\xi]}(q_1, t)\, \psi_{[\xi]}(q_2, t)^*$$ (37)

over pure states with wave functions $\psi_{[\xi]}(q, t)$ defined by

$$\psi_{[\xi]}(q'', t'') = \int dq' \, G_{[\xi]}(q'', t''; q', t') \, \psi(q', t'). \tag{38}$$

According to the expression of the Green function $G_{[\xi]}$, the integral statement for $\psi_{[\xi]}(q, t)$ can be transformed into an operatorial expression by the usual slicing procedure

$$|\psi_{[\xi]}(t'')\rangle = \hat{T} \exp \left(\int_{t'}^{t''} dt \left[-\frac{i}{\hbar} \hat{H}(\hat{p}, \hat{q}, t) - \kappa(t)[\hat{A}(\hat{p}, \hat{q}, t) - a_{[\xi]}(t)]^2 \right. \right.$$
$$\left. \left. + \sqrt{\kappa(t)}[\hat{A}(\hat{p}, \hat{q}, t) - a_{[\xi]}(t)] \, \xi(t) \right] \right) |\psi_{[\xi]}(t')\rangle \tag{39}$$

where $\hat{T}$ means chronological ordering. By introducing the Wiener process $dw(t) = \xi(t) \, dt$ with Ito algebra $dw(t)^2 = dt$ and $dw(t)^{2+n} = 0$ $(n > 0)$, Eq. (39) is recognized as the formal solution of the following Ito stochastic differential equation [43]

$$d |\psi_{[\xi]}(t)\rangle = \left[-\frac{i}{\hbar} \hat{H}(\hat{p}, \hat{q}, t) - \frac{1}{2} \kappa(t)[\hat{A}(\hat{p}, \hat{q}, t) - a_{[\xi]}(t)]^2 \right] |\psi_{[\xi]}(t)\rangle \, dt$$
$$+ \sqrt{\kappa(t)}[\hat{A}(\hat{p}, \hat{q}', t) - a_{[\xi]}(t)] |\psi_{[\xi]}(t)\rangle \, dw(t). \tag{40}$$

In (40) we still have to specify the function $a_{[\xi]}(t)$. According to the decomposition (37), the conservation of the trace of the reduced density matrix operator gives

$$1 = \int d[\xi] \langle \psi_{[\xi]}(t) | \psi_{[\xi]}(t) \rangle. \tag{41}$$

Since the Gaussian measure $d[\xi]$ is normalized, up to a zero-average fluctuation we have $\langle \psi_{[\xi]}(t) | \psi_{[\xi]}(t) \rangle = 1$. The normalization of the state $|\psi_{[\xi]}(t)\rangle$ uniquely determines the expression of $a_{[\xi]}(t)$. Indeed, by evaluating the Ito differential

$$d \langle \psi_{[\xi]}(t) | \psi_{[\xi]}(t) \rangle = 2 \sqrt{\kappa(t)} \langle \psi_{[\xi]}(t)| \hat{A}(t) - a_{[\xi]}(t) |\psi_{[\xi]}(t)\rangle \, dw(t) + \mathcal{O}(dt^{3/2}) \tag{42}$$

we see that the norm of the state $|\psi_{[\xi]}(t)\rangle$ is conserved only if we impose

$$a_{[\xi]}(t) = \langle \psi_{[\xi]}(t)| \hat{A}(t) |\psi_{[\xi]}(t)\rangle. \tag{43}$$

In this case the stochastic differential equation (40) becomes the nonlinear quantum state diffusion equation proposed in [16, 20, 21].

Unlike the effective wave equation (28), the quantum state diffusion equation (40) is suitable for the *a priori* analysis of the selective processes but awkward when dealing with the *a posteriori* analysis. Indeed, in the latter case one should guess the noise realization $\xi(t)$ giving rise, according to (31) and (43), to the known value of

the measurement result $a(t)$. As in the decomposition of the reduced density matrix in terms of the pure states $|\psi_{[a]}(t)\rangle$, the nonselective measurement result (3) and variance (4) can be expressed in terms of the pure states $|\psi_{[\xi]}(t)\rangle$ by

$$\overline{a(t)} = \int d[\xi] \langle \psi_{[\xi]}(t)| \hat{A}(t) |\psi_{[\xi]}(t)\rangle \qquad (44)$$

and

$$\Delta a(t)^2 = \int d[\xi] \langle \psi_{[\xi]}(t)| (\hat{A}(t) - \overline{a(t)})^2 |\psi_{[\xi]}(t)\rangle, \qquad (45)$$

respectively.

As fifth and final step, the equivalence loop of Fig. 1 is closed by regaining the starting differential equation for the reduced density matrix operator

$$\hat{\rho}(t) = \overline{|\psi_{[\xi]}(t)\rangle \langle \psi_{[\xi]}(t)|} \qquad (46)$$

where overlining means functional integration with measure $d[\xi]$. By using Ito algebra we get

$$d\hat{\rho}(t) = \overline{d\,|\psi_{[\xi]}(t)\rangle \langle \psi_{[\xi]}(t)| + |\psi_{[\xi]}(t)\rangle\, d\,\langle \psi_{[\xi]}(t)| + d\,|\psi_{[\xi]}(t)\rangle\, d\,\langle \psi_{[\xi]}(t)|}$$

$$= \left(-\frac{i}{\hbar} [\hat{H}(t), \hat{\rho}(t)] - \frac{1}{2}\kappa(t)[\hat{A}(t), [\hat{A}(t), \hat{\rho}(t)]] \right) dt + \mathcal{O}(dt^{3/2}) \qquad (47)$$

which is Eq. (1) with the previously discussed choice for the Lindblad operators.

III. Influence Functional from an Informational Environment

Deeper insight into the problem of measurement in quantum mechanics can be gained by completing the formal discussion of the previous section with general but explicit models for a measurement process. For this purpose the influence functional approach is a quite appropriate one and some attempts have already been done in this direction [29, 37, 38, 44, 45]. Here we discuss the measurement process in terms of an informational environment made of particles linearly interacting with the measured system and in contact with a heat reservoir at fixed temperature. Each particle selectively measures the evolution of the system while the collection of them represents an apparatus performing a nonselective measurement. This model has two nice features. Firstly, it is simple from a mathematical point of view and the corresponding influence functional can be evaluated exactly [40]. Secondly, very clear and controllable approximations can be made on the influence functional in order to reduce it to the formal expression given in the previous section. These approximations are physically related to the fact that the informational environment can be defined as a many-body system in which each particle, still interacting

quantum mechanically with the measured system, behaves classically from the point of view of the observer. Only in this case the decision of the observer to look or not to look at the pointer of the instrument does not influence the result of the measurement itself [39].

We start by considering the Hamiltonian of a measured system interacting with an environment

$$H_{tot} = H + H_{env} \tag{48}$$

where $H = H(p, q, t)$ is the Hamiltonian of the measured system with phase-space coordinates p and q and H_{env} is the Hamiltonian of the interacting environment is made by different sets of particles with mass M and phase-space coordinates Q_{vn} and P_{vn}, where v labels the sets and n the particles in each set. The particles of each set v interact linearly with an observable $A_v(p, q, t)$ of the system through a function $\lambda_v(t)$ which transduces a displacement of the observables A_v into displacements of the coordinates Q_{vn}

$$H_{env} = \sum_{vn} \left(\frac{P_{vn}^2}{2M} + \frac{M\omega_{vn}^2}{2} [Q_{vn} - \lambda_v(t) A_v(p, q, t)]^2 \right). \tag{49}$$

The interaction of the nth particle of the environment with the vth observable is characterized by a proper angular frequency ω_{vn}. In terms of spectral response we can say that each particle measures the Fourier component of $\lambda_v(t)A_v(p, q, t)$ at its proper angular frequency ω_{vn}.

We assume that at time t' the system is described by the density matrix $\rho(q_1', q_2', t')$ and each particle of the environment is at thermal equilibrium with temperature T. Since the equilibrium value for the coordinates Q_{vn} depends on the coordinates p and q, the assumption of thermal equilibrium at time t' implies a correlation between the environment and the system at the same instant of time [46]. The density matrix for the total system at time t' can be written as

$$\rho_{tot}(q_1', Q_1', q_2', Q_2', t') = \rho(q_1', q_2', t') \, \rho_{env}(\delta Q_1', \delta Q_2', t') \tag{50}$$

where

$$\rho_{env}(\delta Q_1', \delta Q_2', t') = \prod_{vn} \sqrt{\frac{M\omega_{vn}}{\pi h} \tanh\left(\frac{h\omega_{vn}}{2k_B T}\right)}$$

$$\times \exp\left(-\frac{M\omega_{vn}}{2h} \left(\frac{\delta Q_{1vn}'^2 + \delta Q_{2vn}'^2}{\tanh(h\omega_{vn}/k_B T)} + \frac{2\delta Q_{1vn}'\delta Q_{2vn}'}{\sinh(h\omega_{vn}/k_B T)} \right) \right) \tag{51}$$

and

$$\delta Q_{vn}' = Q_{vn}' - \tfrac{1}{2}\lambda_v(t')[A_v(p_1', q_1', t') + A_v(p_2', q_2', t')]. \tag{52}$$

The initial displacements $\delta Q_{vn}'$ are uniquely expressed in terms of the values assumed by the measured observable at the initial points p_1', q_1', t' and p_2', q_2', t' by requiring translational and time reversal invariance [46].

The steps required to get the expression of the influence functional are standard [36, 40]. At time $t'' > t'$ the reduced density matrix of the system, obtained by tracing out the coordinates of the environment in the total density matrix

$$\rho(q_1'' q_2'', t'') = \int dQ'' \, \rho_{tot}(q_1'', Q'', q_2'', Q'', t''), \tag{53}$$

can be obtained by propagating the initial density matrix $\rho(q_1', q_2', t')$

$$\rho(q_1'' q_2'', t'') = \int dq_1' \, dq_2' \, G(q_1'', q_2'', t''; q_1', q_2', t') \, \rho(q_1', q_2', t') \tag{54}$$

by the two-point Green function

$$G(q_1'', q_2'', t''; q_1', q_2', t') = \int d[p_1] \, d[q_1]_{q_1', t'}^{q_1'', t''} \int d[p_2] \, d[q_2]_{q_2', t'}^{q_2'', t''}$$

$$\times \exp\left(\frac{i}{\hbar} S[p_1, q_1] - \frac{i}{\hbar} S[p_2, q_2]\right) F[p_1, q_1, p_2, q_2] \tag{55}$$

where $S[p, q]$ is given by (14) and the influence functional F is

$$F[p_1, q_1, p_2, q_2] = \int dQ'' \int dQ_1' \, dQ_2' \int d[P_1] \, d[Q_1]_{Q_1', t'}^{Q'', t''} \int d[P_2] \, d[Q_2]_{Q_2', t'}^{Q'', t''}$$

$$\times \exp\left(\frac{i}{\hbar} S_{env}[P_1, Q_1, A_1] - \frac{i}{\hbar} S_{env}[P_2, Q_2, A_2]\right)$$

$$\times \rho_{env}(\delta Q_1', \delta Q_2', t') \tag{56}$$

with

$$S_{env}[P, Q, A] = \int_{t'}^{t''} dt \left(\sum_{vn} P_{vn} \dot{Q}_{vn} - H_{env}(P, Q - \lambda A, t)\right). \tag{57}$$

In the above formulas the notation $P, Q, \delta Q$ is a shortening for $\{P_{vn}\}$, $\{Q_{vn}\}$, $\{\delta Q_{vn}\}$ and λ, A for $\{\lambda_v\}$, $\{A_v\}$. The integrations in (56) are Gaussian and can be performed giving

$$F[p_1, q_1, p_2, q_2] = \prod_v \exp\left(-\int_{t'}^{t''} dt \int_{t'}^{t} ds \, \lambda_v(t)[A_v(p_1, q_1, t) - A_v(p_2, q_2, t)]\right.$$

$$\times (\alpha_v(t-s) \, \lambda_v(s)[A_v(p_1, q_1, s) - A_v(p_2, q_2, s)]$$

$$\left. + i\beta_v(t-s) \frac{d}{ds} (\lambda_v(s)[A_v(p_1, q_1, s) + A_v(p_2, q_2, s)]))\right) \tag{58}$$

where the two kernels

$$\alpha_v(t-s) = \frac{M}{2h} \sum_n \omega_{vn}^3 \coth\left(\frac{h\omega_{vn}}{2k_BT}\right) \cos[\omega_{vn}(t-s)] \tag{59}$$

and

$$\beta_v(t-s) = \frac{M}{2h} \sum_n \omega_{vn}^2 \cos[\omega_{vn}(t-s)] \tag{60}$$

describe fluctuation and dissipation phenomena, respectively [40].

Now we turn the attention to the requirement that the environment is informational, i.e., classical with respect to the observer so that the readout of information through the coordinates Q has an objective value. Since a classical system is one whose quantized structure can not be appreciated we must impose the thermal fluctuations to be large in comparison to the quanta of the environment

$$k_BT \gg h\omega_{vn}. \tag{61}$$

In this case we have

$$\alpha_v(t,s) = \frac{Mk_BT}{h^2} \sum_n \omega_{vn}^2 \cos[\omega_{vn}(t-s)]. \tag{62}$$

Memory effects in the fluctuation and dissipation kernels are an inessential complication which can be avoided by assuming an ensemble of particles in the environment with a continuous spectrum of frequencies. If, for simplicity, we choose the same frequency density $dN_v/d\omega = \Omega/\pi\omega^2$ for each set v we get

$$\sum_n \omega_{vn}^2 \cos[\omega_{vn}(t-s)] \simeq \int_0^\infty d\omega\, \frac{dN_v}{d\omega}\, \omega^2 \cos[\omega(t-s)] = \Omega\delta(t-s). \tag{63}$$

Of course, the above assumption of continuous frequency spectrum implies that the condition (61) can not be satisfied in the whole frequency range $[0, +\infty[$ by a finite temperature. Therefore, we should assume the condition (61) to be valid only in a finite range $[0, \omega_{max}]$ which contains the most significant part of the Fourier spectrum of $\lambda_v(t) A_v(p, q, t)$ and is relevant in monitoring the measured observables. In this case the condition (61) allows us to neglect the dissipation term, proportional to (60), with respect to the fluctuation one, proportional to (62). In conclusion, the influence functional for an informational environment can be written as

$$F[p_1, q_1, p_2, q_2] = \prod_v \exp\left(-\tfrac{1}{2}\int_{t'}^{t''} dt\, \kappa_v(t)[A_v(p_1, q_1, t) - A_v(p_2, q_2, t)]^2\right) \tag{64}$$

with

$$\kappa_v(\mathrm{t}) = \lambda_v(\mathrm{t})^2 \frac{2M\Omega k_B T}{\hbar^2}. \tag{65}$$

Equations (64) and (65) generalize to more than one monitored quantity the expression of the functional already introduced in (15) for nonselective measurements. Note that $\kappa_v(t)$ can be written as $\kappa_v(t) = \lambda_v(t)^2/\sigma^2\tau$ where $\sigma^2 = \hbar/2M\Omega$ and $\tau = \hbar/k_B T$. The two parameters σ and τ can be made indefinitely small by increasing the density of meters (particles) in the informational environment proportional to Ω, or their temperature T, respectively. These characteristic length and time play the role of similar parameters introduced *ad hoc* in the spontaneous localization [10] and in the restricted path-integral [28] approaches.

We close this section by evaluating the measurement result arising as a readout from the informational environment and its corresponding variance. The observer reads the result of the nonselective measurement of the vth observable by looking at a pointer which responds to the coordinates of the measurement apparatus. This response is a weighted sum over the proper frequencies of the informational environment and is characterized by a normalized response function ζ_{vn} such that the pointer displacement and its variance are

$$P_v(t) = \sum_n \zeta_{vn} \overline{Q_{vn}(t)} \tag{66}$$

$$\Delta P_v(t)^2 = \sum_n \zeta_{vn} \Delta Q_{vn}(t)^2 \tag{67}$$

where

$$\overline{Q_{vn}(t)} = \mathrm{Tr}[\hat{Q}_{vn}\hat{\rho}_{tot}(t)] \tag{68}$$

$$\Delta Q_{vn}(t)^2 = \mathrm{Tr}[(\hat{Q}_{vn} - \overline{Q_{vn}(t)})^2 \hat{\rho}_{tot}(t)]. \tag{69}$$

Due to the condition (61) the thermal relaxation time of the measurement apparatus turns out to be much smaller than the characteristic timescales of the measured system. This allows us to use an adiabatic approximation for the total density matrix operator and factorize it as

$$\hat{\rho}_{tot}(t) \simeq \hat{\rho}(t)\,\hat{\rho}_{env}(t) \tag{70}$$

where $\hat{\rho}(t)$ is the reduced density matrix operator of the system which takes into account the influence of the informational environment and $\hat{\rho}_{env}(t)$ is the density matrix operator of the informational environment at thermal equilibrium around the instantaneous value of the measured observables. In the representation of the environment coordinates $\hat{\rho}_{env}(t)$ has matrix elements

$$\langle Q_1| \hat{\rho}_{env}(t) |Q_2\rangle = \prod_{vn} \sqrt{\frac{M\omega_{vn}}{\pi h} \tanh\left(\frac{h\omega_{vn}}{2k_B T}\right)}$$

$$\times \exp\left(-\frac{M\omega_{vn}}{2h}\left(\frac{\delta\hat{Q}_{1vn}(t)^2 + \delta\hat{Q}_{2vn}(t)^2}{\tanh(h\omega_{vn}/k_B T)} + \frac{2\delta\hat{Q}_{1vn}(t)\,\delta\hat{Q}_{2vn}(t)}{\sinh(h\omega_{vn}/k_B T)}\right)\right) \tag{71}$$

which are operators with respect to the system coordinates through the displacements

$$\delta\hat{Q}_{vn}(t) = Q_{vn}(t) - \tfrac{1}{2}\lambda_v(t)[\hat{A}_v(\hat{p}_1, \hat{q}_1, t) + \hat{A}_v(\hat{p}_2, \hat{q}_2, t)]. \tag{72}$$

In this case the traces in (68) and (69) contain Gaussian integrals over the environment coordinates which can be performed giving

$$\mathrm{Tr}[\hat{Q}_{vn}\hat{\rho}_{tot}(t)] = \lambda_v(t)\,\mathrm{Tr}[\hat{A}_v(t)\,\hat{\rho}(t)] \tag{73}$$

$$\mathrm{Tr}[\hat{Q}_{vn}^2\hat{\rho}_{tot}(t)] = \frac{h}{2M\omega_{vn}}\coth\left(\frac{h\omega_{vn}}{2k_B T}\right) + \lambda_v(t)^2\,\mathrm{Tr}[\hat{A}_v(t)^2\,\hat{\rho}(t)]. \tag{74}$$

In the limit (61) and for a continuous spectrum of frequencies with the previously chosen density we have

$$P_v(t) = \lambda_v(t)\,\overline{a_v(t)} \tag{75}$$

$$\Delta P_v(t)^2 = \int d\omega\, \zeta_v(\omega) \frac{dN_v}{d\omega} \frac{k_B T}{M\omega^2} + \lambda_v(t)^2\,\Delta a_v(t)^2 \tag{76}$$

where $\overline{a_v(t)}$ and $\Delta a_v(t)^2$ are given by (3) and (4), respectively.

Up to the transduction factor $\lambda_v(t)$ and $\lambda_v(t)^2$, respectively, Eqs. (75) and (76) are the result of the measured observable A_v and its variance read from the pointer. The result of the measurement is the nonselective outcome of the observable A_v obtained from measurement quantum mechanics. The measurement variance is the sum of a classical variance associated to the measurement apparatus and the quantum variance associated to the measured system. The nonselective result and variance (75) and (76) can be expressed in terms of selective processes. The decomposition of $\overline{a_v(t)}$ and $\Delta a_v(t)^2$ has already been discussed in the previous section. Concerning the decomposition of the variance associated to the measurement apparatus we note that the nonselective measurement performed by the informational environment is achieved by summing over the frequencies of the particles in the environment. Therefore, the contribution to the measurement variance given by the selective process at frequency ω is $k_B T/M\omega^2$ as expected from the equipartition theorem.

Equation (76) sets the definition of classical and quantum measurements. The first ones are those characterized by a dominance of the variance associated to the

measurement apparatus, i.e., by a dominance of the thermal (or Brownian) noise. In the second ones the quantum variance of the measured system is larger than the thermal noise. This second case promises a richer variety of experimental scenarios because the measurement variance depends on the preparation of the measured system and/or the strength of the coupling to the measurement apparatus.

4. EXPERIMENTS ON QUANTUM ZENO EFFECT

In many experimental situations one is dealing with an average of measurements on an individual quantum system each time prepared in the same initial state or a single measurement on an ensemble of independent identical quantum systems with the same initial conditions. In both cases or in a combination of them averaged measurement results, instead of individual measurement results, are actually registered as outcome of the experiment. According to the discussion of section II we have two ways for theoretically reproducing the outcome of such an experiment. We can consider *a priori* selective measurements and obtain the experiment outcome by averaging the corresponding selective results in the sense of Eq. (44). We can also consider a nonselective measurement and evaluate the experiment outcome directly from Eq. (3). The choice between the two methods may depend on the particular problem one is faced to. The method based on a nonselective process is more direct but it is based upon the solution of a master equation for the reduced density matrix operator which could be much more difficult than solving many times the the diffusion state equation (40) and averaging the selective measurement results. In this section we will deal with an experimental situation which can be described by a simple 2 by 2 density matrix. The choice of the direct method based on nonselective measurements is, therefore, the natural one.

Let us consider a system with time-independent Hamiltonian and discrete energy spectrum

$$\hat{H} |n\rangle = E_n |n\rangle. \tag{77}$$

The evolution of the system subjected both to an external time-dependent perturbation $\hat{V}(t)$ and to a continuous nonselective measurement of the observable represented by the operator $\hat{A}$ is given by

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar} [\hat{H} + \hat{V}(t), \hat{\rho}(t)] - \frac{1}{2} \kappa(t)[\hat{A}, [\hat{A}, \hat{\rho}(t)]]. \tag{78}$$

An interesting situation is attained in the case the external perturbation $\hat{V}$ stimulates the system to make transitions among the unperturbed levels n and the occupancy of some of these levels is measured. At what extent does the measurement disturb the stimulated transitions? Eventually, inhibition of the stimulated transitions due to the occupancy measurement occurs and one has an example of what is called quantum Zeno effect [47–53].

From a theoretical point of view a two-level system is the simplest one for studying the interplay between the effects of the measurement process and of the external perturbation. This situation has been also experimentally investigated [41]. An ensemble of about $5000\,^9Be^+$ ions was stored in a Penning trap. Two hyperfine levels of the ground state of $^9Be^+$, created by a static magnetic field and hereafter called levels 1 and 2, were driven by a radiofrequency, resonant between levels 1 and 2, turned on for $T = 256$ ms. The amplitude of the radiofrequency was adjusted to make the initially vanishing occupancy of level 2 to be unity at time T in absence of other disturbances. During the radiofrequency pulse, n optical pulses of length $\tau = 2.4$ ms and frequency equal to the transition frequency between level 1 and a third level 3 were also applied. The number of photons emitted in the spontaneous transition $3 \to 1$, the transition $3 \to 2$ being forbidden, was roughly proportional to the occupancy of level 1. The optical pulses acted, therefore, as a measurement of the occupancy of level 1. The occupancy at time T of level 1 was observed to be frozen near its initial unity value, i.e., the stimulated transition $1 \to 2$ in the period T to be inhibited, proportionally to the number n of optical pulses.

Before trying a direct interpretation of the experiment [41], let us consider the case of a continuous measurement of the occupancy of level 1 in a two-level system subjected to stimulated transitions. In the representation of the unperturbed Hamiltonian $\hat{H}$ where $\rho_{nm} = \langle n| \hat{\rho} |m \rangle$ we assume a perturbation $\hat{V}$ with matrix elements $V_{11} = V_{22} = 0$ and $V_{12} = V_{21}^* = V_0 e^{i\omega(t - t_0)}$ with V_0 real and $\omega = (E_2 - E_1)/\hbar + \delta\omega$. In the same representation the matrix elements of the measured occupancy of level 1 are $A_{11} = 1$ and $A_{12} = A_{21} = A_{22} = 0$. The master equation for the reduced density matrix operator then gives

$$\dot{\rho}_{11}(t) = -\frac{i}{\hbar} [V_{12}\rho_{21}(t) - \rho_{12}(t) V_{21}]$$

$$\dot{\rho}_{22}(t) = -\frac{i}{\hbar} [V_{21}\rho_{12}(t) - \rho_{21}(t) V_{12}] \tag{79}$$

$$\dot{\rho}_{12}(t) = \left[-\frac{i}{\hbar}(E_1 - E_2) - \frac{\kappa}{2} \right] \rho_{12}(t) - \frac{i}{\hbar} [V_{12}\rho_{22}(t) - \rho_{11}(t) V_{12}]$$

and $\rho_{21}(t) = \rho_{12}(t)^*$. The measurement coupling κ is assumed constant. By assuming and subtracting the first two equations we find

$$\dot{\rho}_{11}(t) + \dot{\rho}_{22}(t) = 0$$

$$\dot{\rho}_{22}(t) - \dot{\rho}_{11}(t) = \frac{4}{\hbar} \text{Im}(\rho_{12}(t) V_{21}) \tag{80}$$

$$\dot{\rho}_{12}(t) = \left[-\frac{i}{\hbar}(E_1 - E_2) - \frac{\kappa}{2} \right] \rho_{12}(t) - \frac{i}{\hbar} V_{12}(\rho_{22}(t) - \rho_{11}(t)).$$

The first equation is the conservation of the trace of the reduced density matrix operator in the case of a nonselective measurement, $\rho_{11}(t) + \rho_{22}(t) = 1$. By defining

$$\rho_{12}(t) = \exp(i\omega(t - t_0))[\alpha(t) + i\beta(t)] \qquad \rho_{22}(t) - \rho_{11}(t) = \gamma(t) \qquad (81)$$

with α, β and γ real, the other two equations give

$$\dot{\alpha}(t) = -\frac{\kappa}{2}\alpha(t) + \delta\omega\beta(t)$$

$$\dot{\beta}(t) = -\frac{\kappa}{2}\beta(t) - \frac{\omega_R}{2}\gamma(t) - \delta\omega\alpha(t) \qquad (82)$$

$$\dot{\gamma}(t) = 2\omega_R\beta(t)$$

where we have introduced the Rabi angular frequency

$$\omega_R = 2\frac{V_0}{h}. \qquad (83)$$

The above system has a simple solution for $\delta\omega = 0$ (resonance) and in this case we get

$$\rho_{11}(t) = \frac{1}{2} - \frac{1}{2}e^{-(1/4)\kappa t}\left[(\rho_{22}(0) - \rho_{11}(0))\left(\cos(wt) + \frac{\kappa}{4w}\sin(wt)\right)\right.$$

$$\left. + \mathrm{Im}(\rho_{12}(0)e^{i\omega t_0})\frac{2\omega_R}{w}\sin(wt)\right] \qquad (84)$$

$$\rho_{12}(t) = e^{-(i/\hbar)(E_1 - E_2)t - (1/4)\kappa t - i\omega t_0}\left[\mathrm{Re}(\rho_{12}(0)e^{i\omega t_0})e^{-(1/4)\kappa t} + \mathrm{Im}(\rho_{12}(0)e^{i\omega t_0})\right.$$

$$\times\left(\cos(wt) - \frac{\kappa}{4w}\sin(wt)\right)$$

$$\left. - i(\rho_{22}(0) - \rho_{11}(0))\frac{\omega_R}{2w}\sin(wt)\right] \qquad (85)$$

where

$$w = \sqrt{\omega_R^2 - \tfrac{1}{16}\kappa^2}. \qquad (86)$$

The angular frequency w coincides with the Rabi angular frequency ω_R when $\kappa = 0$. In this case the effect of the measurement disappears and the system oscillates between levels 1 and 2 with angular frequency ω_R. In the opposite limit of strong measurement coupling the frequency w is imaginary and an overdamped regime is

achieved in which transitions are inhibited. The border between the two regimes is at $w = 0$ corresponding to a critical measurement coupling

$$\kappa_{crit} = 4\omega_R = 8 \frac{V_0}{\hbar}. \tag{87}$$

The behavior of the measured occupancy of level 1 which, according to Eq. (3), turns out to be $\rho_{11}(t)$ is shown in Fig. 2 for different values of the measurement coupling starting from initial conditions $\rho_{11}(0) = 1$ and $\rho_{12}(0) = 0$. The transition between the Rabi-like regime and the Zeno-like regime is marked by the disappearing of the oscillatory behavior at $\kappa = \kappa_{crit}$.

The quantum variance associated to the nonselective measurement of level 1 can be evaluated according to Eq. (4) and is $\rho_{11}(t) \rho_{22}(t)$. It vanishes in the limit of strong measurement coupling and oscillates with angular frequency ω_R between 0 and 1/4 in the opposite limit $\kappa \to 0$. At the critical measurement coupling $\kappa = \kappa_{crit}$, after a short transient of the order of ω_R^{-1}, the quantum variance approaches the constant and maximum value $(1/2)^2$.

It is worth to note that an identical behavior of the reduced density matrix operator is obtained if we consider $\hat{H}$ as the measured quantity. In this case Eqs. (84) and (85) still hold with the substitution $\kappa \to \kappa_E (E_2 - E_1)^2$ where κ_E is the measurement coupling associated to the measurement of $\hat{H}$. This observation allows
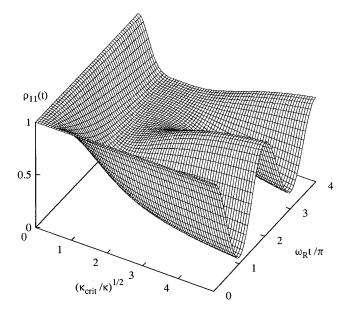

FIG. 2.   Measurement result $\rho_{11}(t)$ during the nonselective measurement of occupancy of level 1 in a two-level system simultaneously driven by a resonant perturbation as a function of the adimensional quantity κ_{crit}/κ. The system is prepared in level 1 at time $t = 0$.

us to compare Fig. 2 with Fig. 1 of Ref. [27] where the quantum Zeno effect for the same two-level system investigated here was analyzed *a posteriori* in the case of a single selective measurement of energy. As expected, quantitative differences between nonselective measurements and single selective measurements are obtained but in both cases Zeno inhibition occurs for strong measurement coupling. The behavior shown in Fig. 2 agrees also with that found in [54, 55] where the quantum Zeno effect is analyzed within the quantum trajectory approach [56].

The discussion relative to Fig. 2 can be made general. In the limit $\kappa \to 0$ the influence of the measurement disappears and ordinary quantum mechanics is recovered. In the limit $\kappa \to \infty$ the influence of the measurement dominates and the average measurement result (3) as well as the corresponding variance (4) are frozen into the values assumed at the beginning of the measurement. In the case of Fig. 2 these values are $\rho_{11}(0) = 1$ and $\rho_{11}(0)\,\rho_{22}(0) = 0$, respectively. Moreover, in the limit $\kappa \to \infty$ the off-diagonal elements of the reduced density matrix operator vanish and a classical behavior is obtained.

In order to recover the results of the experiment [41] the previous analysis for a continuous nonselective measurement with κ constant must be generalized to a series of measurement pulses spaced by intervals of no measurement. According to the experimental procedure we consider n nonselective measurements of the occupancy of level 1 with coupling κ during the intervals $[jT/n - \tau,\ jT/n]$, $j = 1, ..., n$, with $\tau = 2.4$ ms and $T = 256$ ms. During the remaining part of the interval $[0, T]$ the system is subjected only to the radiofrequency perturbation at $\omega = 2\pi \times 320.7$ MHz whose amplitude V_0 is fixed by the condition $\omega_R = \pi/T$ that gives, in absence of measurement, $\rho_{22}(T) = 1$ if $\rho_{22}(0) = 0$. Since the radiofrequency is resonant $\delta\omega = 0$. The time evolution of the reduced density matrix operator corresponding to the described process is obtained by successive iterations of (84) and (85) with the same formulas evaluated for $\kappa = 0$. The theoretical probability for the transition $1 \to 2$ at the end of the interval T

$$P^{th}_{1 \to 2}(n, \kappa) = 1 - \rho_{11}(T) \tag{88}$$

can be compared with the corresponding experimental data $P^{exp}_{1 \to 2}(n)$ available for $n = 1, 2, 4, 8, 16, 32, 64$. The uncertainty for the experimental transition probabilities is estimated to be $\Delta P = 0.02$ [41].

In Fig. 3 we show the sum of the squared differences between the theoretical and experimental transition probabilities normalized to the experimental uncertainties, the χ^2, as a function the phenomenological parameter κ/κ_{crit}. In the same figure we show also the probability $Q(\chi^2 \mid \nu)$ that the observed value for the chi-square should exceed the value $\chi^2(\kappa/\kappa_{crit})$ by chance. In our case the number of degrees of freedom is $\nu = 7 - 1$. The two arrows indicate the chi-square value obtained by the same authors of [41] using a theoretical model based on the instantaneous von Neumann collapse without (higher value) or with (lower value) corrections for the finite duration of the measurement pulses, the effect of optical pumping from level 2 to level 1 and the measured value of ω_R. Depending upon the statistical
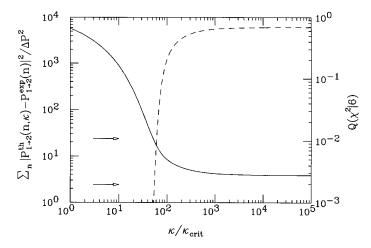
FIG. 3. Behavior of $\chi^2 = \sum_n |P_{1\to2}^{th}(n, \kappa) - P_{1\to2}^{exp}(n)|^2/\Delta P^2$ as a function of κ/κ_{crit} in fitting the experiment [41]. The dashed line is the probability $Q(\chi^2 \mid 6)$ that the chi-square should exceed the value $\chi^2(\kappa/\kappa_{crit})$ by chance. The two arrows indicate the chi-square value obtained in [41] by a model based on instantaneous von Neumann collapse without (higher value) or with (lower value) corrections for finite measurement-pulse duration, optical pumping and measured value of ω_R.

confidence level we adopt, from Fig. 3 we see that values of $\kappa/\kappa_{crit} \gtrsim 10^2$ are required to fit properly the experimental data. The minimum value of χ^2 obtained from measurement quantum mechanics slightly differs from the χ^2 obtained on the basis of the von Neumann postulate including experimental corrections.

It is neither surprising nor exciting that measurement quantum mechanics is able to explain the experimental results of Ref. [41] in terms of a strong Zeno inhibition. Not surprising because measurement quantum mechanics contains more naive approaches to the problem of quantum measurements as the von Neumann postulate, for instance, which was already shown to reproduce the experimental results. Not exciting because strong Zeno inhibition as well as full Rabi oscillations are two trivial extreme regimes. However, measurement quantum mechanics tells us that another interesting and unexplored regime exists. It is the regime which occurs when the measurement coupling is comparable to the critical value. In this case a strong competition between stimulated transitions and measurement inhibition takes place as clearly recognized from Fig. 2 and also from the analysis of the corresponding quantum variance, $\rho_{11}\rho_{22}$, which gets maximum at $\kappa = \kappa_{crit}$. Measurement quantum mechanics suggests us also how to explore this regime. In order to make $\kappa/\kappa_{crit} \sim 1$ we should decrease κ and/or increase κ_{crit} with respect to the values used in [41]. According to the discussion of section III κ can be decreased by reducing the density of particles in the informational environment. In the experiment [41] the "particles" of the informational environment are the excited modes of the photon vacuum. Their density can be reduced by lowering the intensity of the optical radiation acting as a measurement probe. On the other

hand, Eq. (87) shows that κ_{crit} can be increased by means of the amplitude V_0 of the perturbation which stimulates the Rabi oscillations. In the experiment [41] this amounts to increase the intensity of the radiofrequency radiation. Of course, by increasing V_0 the Rabi angular frequency increases too so that the condition $\omega_R = \pi/T$ used in [41] can be maintained only by decreasing the period T and, consequently, the duration of the measurement pulses. The possibility to vary the strength of the coupling is also related to the opportunity of looking for decoherence effects in the same experiment. Indeed, from Eq. (85) we see that the off-diagonal elements of the density matrix vanish exponentially with time constant $\tau_{dec} = 4/\kappa$. For an appropriate measurement coupling, i.e., a proper intensity of the optical radiation, τ_{dec} could be experimentally accessible by optical homodyne tomography techniques [57–59] and decoherence phenomena could be investigated.

5. Conclusions

Measurement quantum mechanics has been introduced by showing the equivalence among formalisms developed with the aim of including the effect of the measurement on the dynamics of a quantum system. The theory contains one parameter, the measurement coupling, for each measured observable.

The measurement couplings can be expressed in terms of the detailed properties of a measurement apparatus which extracts information from the measured system in objective way, i.e., independently of the presence of an observer looking at the pointer of the apparatus. The definition of quantum and classical measurements then emerges naturally according to the dominance, in the uncertainty of the pointer, of the classical fluctuations due to the apparatus or of the quantum fluctuations due to the measured system.

Alternatively, one can introduce the measurement couplings as phenomenological parameters to be inferred from comparison with experimental data, as in the example of the quantum Zeno effect.

Ordinary quantum mechanics is obtained as a limit case for vanishing measurement couplings. In the opposite limit of infinite measurement couplings the dynamics of the measured system is frozen and classical. Both limits are not interesting, the first one because there is no measurement and the second one because there is no quantum dynamics to measure. Interesting physics is in between and measurement quantum mechanics is a tool for designing experiments in this intermediate regime.

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