PERTURBATIVE CRITERIA FOR THE ERGODICITY OF INTERACTING DISSIPATIVE QUANTUM LATTICE SYSTEMS

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Abstract. We introduce a class of quantum Markov semigroups describing the evolution of interacting quantum lattice systems, specified either as generic qudits or as fermions. The corresponding generators, which include both conservative and dissipative evolutions, are given by the superposition of local generators in the Lindblad form. Under general conditions, we show that the associated infinite volume dynamics is well defined and can be obtained as the strong limit of the finite volume dynamics. By regarding the interacting evolution as a perturbation of a non-interacting dissipative dynamics, we further obtain a quantitative criterion that yields the ergodicity of the quantum Markov semigroup together with the exponential convergence of local observables. The analysis is based on suitable a priori bounds on the resolvent equation which yield quantitive estimates on the evolution of local observables.

1. Introduction

A *quantum Markov semigroup* (QMS) is a strongly continuous contraction semigroup $(\mathcal{P}_t)_{t\geq 0}$ of completely positive operators on a C^*-algebra $\mathcal{A}$. It has become the basic modelling tool to describe the non-unitary evolution of open quantum systems, the typical example being the interaction with thermal baths. The generator of a QMS is commonly referred to as the Lindblad generator. Both from a conceptual and an applied viewpoint, the ergodicity properties of QMS are particularly relevant. We refer to [2, 7] for a general overview.

Under general conditions, finite dissipative quantum systems admit a unique stationary state π. As in the case of classical Markov semigroups, a most relevant issue is to provide quantitative estimates on the speed of convergence to the stationary state. More precisely, given an initial state μ one would like to deduce the exponential convergence of its evolution $\mu\mathcal{P}_t$ to π; a natural distance to quantify this convergence is the trace distance, the non-commutative counterpart of the total variation distance. By the quantum Pinsker inequality, see e.g. [23, Thm.11.9.5], the convergence in trace distance can be deduced from the decay of the quantum relative entropy of $\mu\mathcal{P}_t$ with respect to π. For reversible QMS, the latter issue has been recently pursued with different perspectives [6, 8, 9, 13, 24]. In particular, in [8] the exponential decay of entropy is deduced for both the Bose and Fermi Ornstein-Uhlenbeck semigroups that describe the evolution of non-interacting bosons and fermions. As discussed in [5], the exponential convergence in trace distance can also be deduced by spectral methods.

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The present purpose is to investigate the ergodic properties of QMS corresponding to infinite quantum lattice systems. We refer e.g. to [4, 20] for some physically relevant examples. According to the standard framework, a quantum lattice system is described by the so-called *quasilocal* C^*-algebra $\mathcal{A}$. We consider here both the cases in which $\mathcal{A}$ describes generic qudits and fermions. While the analysis of Heisenberg evolutions on $\mathcal{A}$ has been widely investigated, we are not aware of mathematical results concerning dissipative evolutions on $\mathcal{A}$.

As in the Heisenberg case, the natural choice for the generator is given by the superposition of translation covariant local generators. In other words we are led to consider generators on $\mathcal{A}$ of the form

$$\mathcal{L} = \sum_{X \subset\subset \mathbb{Z}^d} L_X, \tag{1.1}$$

where the sum is carried over the finite subsets of $\mathbb{Z}^d$ and the local Lindblad generator L_X acts on $\mathcal{A}_X$, the subalgebra of the X-local operators. In the simplest and most relevant case, the family $\{L_X\}_{X \subset\subset \mathbb{Z}^d}$ is translation covariant and has finite range. For each $X \subset\subset \mathbb{Z}^d$ the C^*-algebra $\mathcal{A}_X$ is finite dimensional and L_X can be prescribed according to the Lindblad-Gorini-Kossakowski-Sudarshan structure theorem [11, 19]. On the other hand, the right-hand side of (1.1) defines an unbounded operator on the C^*-algebra $\mathcal{A}$. A preliminary issue is thus to show that $\mathcal{L}$ generates a QMS on $\mathcal{A}$. This is the dissipative counterpart to the existence of the Heisenberg flow for quantum lattice systems, see e.g. [22, Thm.7.6.2]. Defining $\mathcal{L}$ on a suitable dense domain, we here deduce sufficient conditions on $\{L_X\}_{X \subset\subset \mathbb{Z}^d}$, holding in the translation covariant finite range case, which imply that the graph norm closure of $\mathcal{L}$ generates a QMS $(\mathcal{P}_t)_{t \geq 0}$ on $\mathcal{A}$.

We next turn to the discussion of the ergodic properties of the QMS $(\mathcal{P}_t)_{t \geq 0}$ generated by (1.1). By a soft compactness argument, $(\mathcal{P}_t)_{t \geq 0}$ has at least a stationary state; on the other hand, for instance by considering classical stochastic Ising models as QMS, it is straightforward to exhibit examples for which the stationary state is not unique. If the interaction is small, we expect uniqueness of the stationary state and exponential convergence of local observables.

We deduce a perturbative criterion for the above conclusion. We write $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ where $\mathcal{L}_0$ describes the evolution of non-interacting qudits, i.e. it has the form

$$\mathcal{L}_0 = \sum_{x \in \mathbb{Z}^d} L_x^0,$$

for suitable translation covariant generators L_x^0 that act on $\mathcal{A}_{\{x\}}$. The operator $\mathcal{L}_1$, that takes into account the interaction between the qudits, is then regarded as a perturbation. As discussed in [8, 13, 24], a basic tool in the derivation of strong ergodic properties for the QMS generated by $\mathcal{L}_0$ is the construction of operators $\{E_{x,h}\}$ on $\mathcal{A}$ satisfying the following intertwining relationship with $\mathcal{L}_0$

$$E_{x,h}\mathcal{L}_0 - \mathcal{L}_0 E_{x,h} = -\lambda_h E_{x,h} \tag{1.2}$$

for suitable $\lambda_h > 0$. When $\mathcal{A}$ is the fermionic C^*-algebra and $\mathcal{L}_0$ is the generator of the Fermi Ornstein-Uhlenbeck semigroup, the operators $\{E_{x,h}\}$ have been introduced in [12]. On the other hand, when $\mathcal{A}$ is a C^*-algebra describing generic qudits, we do not specify an explicit form for L_x^0 but we assume that it is self-adjoint with respect to the GNS inner product induced by its unique stationary state. We then construct the operators $E_{x,h}$ from the spectral decomposition of L_x^0; accordingly

$\{\lambda_h\}$ are the eigenvalues of $-L^0_x$. To achieve the perturbative criterion for ergodicity of the QMS generated by $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$, we follow the argument for interacting classical lattice systems in [18, Ch.I]. The key step is the derivation of suitable a priori bounds on the resolvent equation

$$\lambda g = f + \mathcal{L} g, \qquad \lambda > 0, \quad f, g \in \mathcal{A}.$$

By exploiting the intertwining relation (1.2), we derive a quantitative bound on the locality of g in terms of the locality of f. Under a suitable smallness assumption on the commutator $E_{x,h}\mathcal{L}_1 - \mathcal{L}_1 E_{x,h}$, the a priori bound on the resolvent equation yields the uniqueness of the stationary state together with the exponential convergence of local observables. We emphasize that the perturbative criteria here obtained rely neither on the explicit knowledge of stationary states nor on the self-adjointness of the generator.

Whenever the perturbation criterion applies, we can deduce that for any state μ on $\mathcal{A}$ the sequence $\mu\mathcal{P}_t$ converges weakly* to the unique stationary state π as $t \to +\infty$. As in the classical case, it does not appear however possible to obtain a quantitative bound uniform in μ. Indeed, this fails even for non-interacting systems. On the other hand, if we restrict to translation covariant interactions and translation invariant state μ, the above conclusion holds. More precisely, we equip the set of translation invariant states on $\mathcal{A}$ with the specific quantum one-Wasserstein distance w introduced in [15]. We then show that $w(\mu\mathcal{P}_t, \pi)$ decays exponentially uniformly in μ whenever the perturbative criterion holds. This statement appears to be novel even in the context of interacting classical lattice systems.

2. Interacting qudits: results

Given a Banach space $\mathcal{X}$, we denote by $\|\cdot\|_{\mathcal{X}}$ the norm in $\mathcal{X}$ and by $\mathcal{X}'$ the dual of $\mathcal{X}$. The identity operator on $\mathcal{X}$ is denoted by $\mathbb{1}_{\mathcal{X}}$ and the operator norm on the set of bounded linear operators on $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}\to\mathcal{X}}$. In the present setting, given a unital C^*-algebra $\mathcal{A}$, a *Quantum Markov Semigroup* (QMS) $(\mathcal{P})_{t\geq 0}$ on $\mathcal{A}$ is a strongly continuous contraction semigroup on $\mathcal{A}$ (as a Banach space) such that for each $t \in [0,\infty)$ the linear operator $\mathcal{P}_t \colon \mathcal{A} \to \mathcal{A}$ is completely positive and satisfies $\mathcal{P}_t\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the identity in $\mathcal{A}$. If $\mathcal{A}$ is commutative, namely $\mathcal{A}$ is the space of complex valued continuous functions on a compact set endowed with the complex conjugation and the uniform norm, a QMS on $\mathcal{A}$ is a (classical) Markov semigroup.

2.1. **One-qudit unperturbed dynamics.**

Let H be a $(n+1)$-dimensional Hilbert space and A a $(N+1)$-dimensional unital $*$-subalgebra of the C^*-algebra $\mathcal{B}(H)$ of linear operators on H. Since H is finite dimensional, A endowed with the operator norm is closed and therefore it is a C^*-algebra. Let ρ be a faithful state on A, namely a continuous linear functional on A such that $\rho(\mathbb{1}_H) = 1$ and $\rho(aa^*) > 0$ for all $a \in A\setminus\{0\}$. Denote by $\langle\cdot,\cdot\rangle_\rho$ the GNS inner product on A induced by ρ, i.e. $\langle a, b\rangle_\rho = \rho(a^* b)$, $a, b \in A$. Let $(P^0_t)_{t\geq 0}$, be a QMS on A which is self-adjoint with respect to $\langle\cdot,\cdot\rangle_\rho$ and denote by $L_0 \colon A \to A$ its generator. We assume that L_0 has the form

$$L_0 = \sum_{j\in I_0} \left(\ell^{0*}_j[\,\cdot\,,\ell^0_j] + [\ell^{0*}_j,\,\cdot\,]\ell^0_j\right) \tag{2.1}$$

for some finite set I_0 and $\{\ell_j^0\}_{j\in I_0} \subset A$. We refer to [8, Thm.3.1], [1, Thm.3], and [5, Lem. 2.2] for the conditions on $\{\ell_j^0\}_{j\in I_0}$ corresponding to the self-adjointness of L_0 with respect to $\langle\,\cdot\,,\cdot\,\rangle_\rho$.

Since ρ is a stationary state for $(P_t^0)_{t\geq 0}$, the Kadison-Schwarz inequality implies $\langle P_t^0 a, P_t^0 a\rangle_\rho \leq \langle a,a\rangle_\rho$. Therefore $(P_t^0)_{t\geq 0}$ is a self-adjoint strongly continuous contraction semigroup on the Hilbert space $(A, \langle\,\cdot\,,\cdot\,\rangle_\rho)$. Hence $-L_0$ is positive definite with respect to $\langle\,\cdot\,,\cdot\,\rangle_\rho$. Let $\mathbb{1}_H = e_0, e_1, \dots, e_N \in A$ be an orthonormal basis, with respect to $\langle\,\cdot\,,\cdot\,\rangle_\rho$, of eigenvectors of $-L_0$, with eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_N$. In particular,

$$-L_0 = \sum_{h=1}^{N} \lambda_h \, \langle e_h, \,\cdot\,\rangle_\rho \, e_h \,. \tag{2.2}$$

We observe that $\|e_0\|_{H\to H} = 1$ and set $\eta := \max_{h\in\{1,\dots,N\}} \|e_h\|_{H\to H}$.

2.2. **Unperturbed dynamics of qudits.** We denote by $\mathbb{Z}^d$ the standard d-dimensional lattice and by $\mathcal{P}$ the countable family of its finite subsets, $\mathcal{P} := \{X \subset \mathbb{Z}^d : |X| < \infty\}$. Referring to [16] for a detailed exposition, we briefly recall the construction of the Hilbert space and the C^*-algebra describing infinitely many qudits.

Fix an orthonormal basis $\{\omega_i\}_{i=0}^n$ of H, where ω_0 represents the vacuum state. Let Φ be the countable set of functions $\phi : \mathbb{Z}^d \to \{0,1,\dots,n\}$ such that $\phi(x) = 0$ for all but finitely many $x \in \mathbb{Z}^d$. We then let $\mathcal{H}$ be the Hilbert space with orthonormal basis $\{\Omega_\phi\}_{\phi\in\Phi}$. For $\Lambda \subset \mathbb{Z}^d$ we also consider the subspace $\mathcal{H}_\Lambda \subset \mathcal{H}$ spanned by $\{\Omega_\phi\}_{\phi\in\Phi_\Lambda}$, where $\Phi_\Lambda \subset \Phi$ is the subset of functions $\phi \in \Phi$ such that $\phi(x) = 0$ if $x \notin \Lambda$. For $\Lambda \in \mathcal{P}$, we identify $\mathcal{H}_\Lambda \simeq H^{\otimes\Lambda}$ via $\Omega_\phi \mapsto \otimes_{x\in\Lambda}\omega_{\phi(x)}$, $\phi \in \Phi_\Lambda$. Note that the construction of $\mathcal{H}$ depends on the choice of the vacuum ω_0.

Denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of bounded operators on $\mathcal{H}$ and by $\|\cdot\|$ the corresponding operator norm. For $\Lambda \subset \mathbb{Z}^d$ we have the canonical identification $\Phi \simeq \Phi_\Lambda \times \Phi_{\Lambda^c}$, which induces $\mathcal{H} \simeq \mathcal{H}_\Lambda \otimes \mathcal{H}_{\Lambda^c}$. As a consequence we have the canonical embedding $\mathcal{B}(\mathcal{H}_\Lambda) \subset \mathcal{B}(\mathcal{H})$. For $\Lambda \in \mathcal{P}$, we set $\mathcal{A}_\Lambda := A^{\otimes\Lambda} \subset \mathcal{B}(H^{\otimes\Lambda})$, which we identify with the $*$-subalgebra of $\mathcal{B}(\mathcal{H}_\Lambda) \subset \mathcal{B}(\mathcal{H})$, via the above identification $H^{\otimes\Lambda} \simeq \mathcal{H}_\Lambda$. Since $\mathcal{A}_\Lambda$ is finite dimensional, it is a C^*-algebra when equipped with the norm $\|\cdot\|$. Set $\mathcal{A}^0 := \bigcup_{\Lambda\in\mathcal{P}} \mathcal{A}_\Lambda \subset \mathcal{B}(\mathcal{H})$, and let $\mathcal{A}$ be the norm closure of $\mathcal{A}^0$. In particular, $\mathcal{A}$ is a unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. We emphasize that, even in the case $A = \mathcal{B}(H)$, $\mathcal{A}$ is a proper subalgebra of $\mathcal{B}(\mathcal{H})$. For example, for $x \in \mathbb{Z}^d$, consider the translation operator $\tau_x \in \mathcal{B}(\mathcal{H})$, $x \in \mathbb{Z}^d$, defined by

$$\tau_x(\Omega_\phi) := \Omega_{\tau_x\phi}, \quad \text{where} \quad \tau_x\phi := \phi(\,\cdot - x), \qquad \phi \in \Phi. \tag{2.3}$$

As simple to check, τ_x does not belong to $\mathcal{A}$. On the other hand, it induces the automorphism $\mathrm{Ad}_{\tau_x} : \mathcal{A} \to \mathcal{A}$ by $\mathrm{Ad}_{\tau_x}(f) := \tau_x f \tau_{-x}$. In the terminology of [22, §6.2.4], the triple $(\mathcal{A}, \mathbb{Z}^d, \mathrm{Ad}_\tau)$ is the *quasilocal* algebra describing the qudits on $\mathbb{Z}^d$. The set of the states on $\mathcal{A}$ is denoted by $\mathcal{S}$. A state $\pi \in \mathcal{S}$ is *translation invariant* if $\pi\big(\mathrm{Ad}_{\tau_x}(f)\big) = \pi(f)$ for any $x \in \mathbb{Z}^d$ and $f \in \mathcal{A}$; the set of translation invariant states is denoted by $\mathcal{S}_\tau$. A state $\pi \in \mathcal{S}$ is *stationary* for the QMS $(\mathcal{P}_t)_{t\geq 0}$ if $\pi\mathcal{P}_t = \pi$ for any $t \in [0, +\infty)$.

The unperturbed dynamics is next defined by letting each qudit evolve according to the generator L_0 in (2.2). For $x \in \mathbb{Z}^d$ we set $L_x^0 = L_0 \otimes \mathbb{1}_{\mathcal{A}_{\{x\}^c}}$, that is regarded as an operator on $\mathcal{A}$. The unperturbed generator is then informally given by

$$\mathcal{L}_0 = \sum_{x\in\mathbb{Z}^d} L_x^0 = \sum_{\alpha\in\mathcal{I}_0} \left(\ell_\alpha^{0\,*}[\,\cdot\,,\ell_\alpha^0] + [\ell_\alpha^{0\,*},\,\cdot\,]\ell_\alpha^0\right) \tag{2.4}$$

where, recalling (2.1), $\mathcal{I}_0 := \mathbb{Z}^d \times I_0$ and for $\alpha = (x,j)$ the operator ℓ_α^0 corresponds to ℓ_j^0 via the identification $\mathcal{A}_{\{x\}} \simeq A$. For future purposes, we denote by $\chi_0 \colon \mathcal{I}_0 \to \mathbb{Z}^d$ the projection $\chi_0(x,j) = x$. We will show that the right-hand side of (2.4) is well defined on a suitable dense subset of $\mathcal{A}$ and its graph norm closure generates a QMS on $\mathcal{A}$.

Recalling the spectral decomposition (2.2), let $E_{x,h}$, $x \in \mathbb{Z}^d$, $h \in \{0,1,\dots,N\}$, be the linear operators on $\mathcal{A}^0$ acting on monomials as

$$E_{x,h}\big(\otimes_y f_y\big) = \langle e_h, f_x\rangle_\rho \mathbb{1}_H \otimes \big(\otimes_{y\neq x} f_y\big). \tag{2.5}$$

As we prove in Lemma 3.1, $E_{x,h}$ extends to a bounded operator on $\mathcal{A}$. Let also $e_{x,h}$ be the element of $\mathcal{A}_{\{x\}} \simeq A$ corresponding to e_h. Since $\{e_h\}$ is an orthonormal basis of A, for each $x \in \mathbb{Z}^d$ and $f \in \mathcal{A}$

$$f = \sum_{h=0}^N (E_{x,h} f) e_{x,h}. \tag{2.6}$$

Introduce the seminorm $|||\cdot|||$ on $\mathcal{A}^0$ by setting

$$|||f||| := \sum_{x\in\mathbb{Z}^d} \sum_{h=1}^N \|E_{x,h} f\|, \tag{2.7}$$

where we emphasize that $E_{x,0}$ does not appear on the right-hand side. We interpret $\sum_{h=1}^N \|E_{x,h} f\|$ as a measure of the dependence of f on the qudit at site $x \in \mathbb{Z}^d$. In particular, $|||f||| = 0$ if and only if f is a scalar multiple of the identity. Let $\mathcal{A}^1$ be the closure of $\mathcal{A}^0$ with respect to the norm $\|\cdot\| + |||\cdot|||$. Clearly, $\mathcal{A}^0 \subset \mathcal{A}^1 \subset \mathcal{A}$. Let $\ell_1(\mathbb{Z}^d)$ be the Banach space of summable real sequences indexed by $\mathbb{Z}^d$. For $f \in \mathcal{A}^1$, set

$$\delta(f) = \big(\delta_x(f)\big)_{x\in\mathbb{Z}^d} := \Big(\sum_{h=1}^N \|E_{x,h} f\|\Big)_{x\in\mathbb{Z}^d} \in \ell_1(\mathbb{Z}^d)\,, \tag{2.8}$$

so that $|||f||| = \sum_x \delta_x(f) = \|\delta(f)\|_{\ell_1(\mathbb{Z}^d)}$.

2.3. **Dynamics of interacting qudits.** The dynamics of the qudits is defined by an unbounded Lindblad generator on $\mathcal{A}$ given by the sum of local generators. More precisely, we fix a countable set $\mathcal{I}$ and a map $\chi\colon \mathcal{I} \to \mathcal{P}$ such that $|\chi^{-1}(X)| < +\infty$ for any $X \in \mathcal{P}$. We then consider the informal generator on $\mathcal{A}$ given by

$$\mathcal{L} = \sum_{\alpha\in\mathcal{I}} \big(i[k_\alpha,\,\cdot\,] + \ell_\alpha^*[\,\cdot\,,\ell_\alpha] + [\ell_\alpha^*,\,\cdot\,]\ell_\alpha\big) \tag{2.9}$$

for some self-adjoint $k_\alpha \in \mathcal{A}_{\chi(\alpha)}$ and $\ell_\alpha \in \mathcal{A}_{\chi(\alpha)}$. Note that, by setting $K_X = \sum_{\alpha\in\chi^{-1}(X)} k_\alpha$ and

$$L_X = i[K_X,\,\cdot\,] + \sum_{\alpha\in\chi^{-1}(X)} \big(\ell_\alpha^*[\,\cdot\,,\ell_\alpha] + [\ell_\alpha^*,\,\cdot\,]\ell_\alpha\big) \tag{2.10}$$

then $\mathcal{L}$ has the form (1.1). If $A = \mathcal{B}(H)$ so that $\mathcal{A}_X \simeq \mathcal{B}(\mathcal{H}_X)$, by the Lindblad-Gorini-Kossakowski-Sudarshan structure theorem [11, 19], the right-hand side of (2.10) is the general form of a Lindblad generator on $\mathcal{A}_X$.

The family $\{k_\alpha, \ell_\alpha\}_{\alpha\in\mathcal{I}}$ has *finite range* if there exists $R \in [0,\infty)$ such that $k_\alpha = \ell_\alpha = 0$ whenever $\operatorname{diam}(\chi(\alpha)) > R$. The family $\{k_\alpha, \ell_\alpha\}_{\alpha\in\mathcal{I}}$ is *translation covariant* if there exists an action of the abelian group $\mathbb{Z}^d$ on $\mathcal{I}$, denoted by $(x,\alpha) \mapsto x+\alpha$, satisfying $\chi(x+\alpha) = x + \chi(\alpha)$, such that $\mathrm{Ad}_{\tau_x}(k_\alpha) = k_{x+\alpha}$ and $\mathrm{Ad}_{\tau_x}(\ell_\alpha) = \ell_{x+\alpha}$. As we next state, under suitable conditions the operator $\mathcal{L}$ in (2.9) is well defined on the dense subset $\mathcal{A}^1 \subset \mathcal{A}$.

Lemma 2.1. *If*

$$C_0 := 2\eta \sup_{x\in\mathbb{Z}^d} \sum_{\alpha:\,\chi(\alpha)\ni x} \big(\|k_\alpha\| + 2\|\ell_\alpha\|^2\big) < \infty \tag{2.11}$$

then for each $f \in \mathcal{A}^1$ the series defining $\mathcal{L}f$ converges in $\mathcal{A}$ and $\|\mathcal{L}f\| \le C_0 |||f|||$.

2.4. **Main results.** The first result on interacting qudits establishes the existence of the dynamics associated to the generator $\mathcal{L}$ introduced in (2.9). It will be convenient to write it as a perturbation of the dynamics of non-interacting qudits as defined by (2.4). Namely, we let $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ where we recall that both $\mathcal{L}$ and $\mathcal{L}_0$ have been constructed from local Lindblad generators.

This decomposition is achieved by decomposing $\ell_\alpha = \ell^0_{\alpha_0} + \ell^1_{\alpha_0}$ for $\alpha_0 \in \mathcal{I}_0$ and some $\alpha \in \mathcal{I}$. More precisely, fix an injective map $\imath\colon \mathcal{I}_0 \to \mathcal{I}$ such that $\chi_0(\alpha_0) \in \chi(\imath(\alpha_0))$, $\alpha_0 \in \mathcal{I}_0$. Setting $\ell^1_{\alpha_0} = \ell_{\imath(\alpha_0)} - \ell^0_{\alpha_0}$, $\alpha_0 \in \mathcal{I}_0$, we have $\mathcal{L}_1 = \sum_{\alpha\in\mathcal{I}} L^1_\alpha$, where

$$L^1_\alpha = i[k_\alpha,\,\cdot\,] + \begin{cases} {\ell^0_{\alpha_0}}^*[\,\cdot\,,\ell^1_{\alpha_0}] + [{\ell^0_{\alpha_0}}^*,\,\cdot\,]\ell^1_{\alpha_0} & \\ \quad + {\ell^1_{\alpha_0}}^*[\,\cdot\,,\ell^0_{\alpha_0}] + [{\ell^1_{\alpha_0}}^*,\,\cdot\,]\ell^0_{\alpha_0} & \text{if } \alpha = \imath(\alpha_0) \in \imath(\mathcal{I}_0), \\ \quad + {\ell^1_{\alpha_0}}^*[\,\cdot\,,\ell^1_{\alpha_0}] + [{\ell^1_{\alpha_0}}^*,\,\cdot\,]\ell^1_{\alpha_0} & \\ & \\ \ell^*_\alpha[\,\cdot\,,\ell_\alpha] + [\ell^*_\alpha,\,\cdot\,]\ell_\alpha & \text{if } \alpha \notin \imath(\mathcal{I}_0). \end{cases} \tag{2.12}$$

The strength of the perturbation $\mathcal{L}_1$ is measured by

$$M := \sup_{y\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} \theta_{x,y} \tag{2.13}$$

where $\theta_{x,y} := 2N\eta(\theta^0_{x,y} + \theta^1_{x,y})$, in which

$$\begin{aligned}\theta^0_{x,y} := \sum_{\substack{\alpha_0\in\mathcal{I}_0\\ \chi_0(\alpha_0)=y}} \Big(&\big(1+\eta^2\delta_{x,y}\big)\delta_x(k_{\alpha_0}) + 2\big(\eta^2+\delta_{x,y}\big)\Big[\delta_x\big({\ell^0_{\alpha_0}}^*\big)\,\|\ell^1_{\alpha_0}\| + \|{\ell^0_{\alpha_0}}^*\|\,\delta_x\big(\ell^1_{\alpha_0}\big) \\ &+ \delta_x\big({\ell^1_{\alpha_0}}^*\big)\,\|\ell^0_{\alpha_0}\| + \|{\ell^1_{\alpha_0}}^*\|\,\delta_x\big(\ell^0_{\alpha_0}\big) + \delta_x\big({\ell^1_{\alpha_0}}^*\big)\,\|\ell^1_{\alpha_0}\| + \|{\ell^1_{\alpha_0}}^*\|\,\delta_x\big(\ell^1_{\alpha_0}\big)\Big]\Big)\end{aligned}$$

and

$$\theta^1_{x,y} := \sum_{\substack{\alpha\in\mathcal{I}\setminus\imath(\mathcal{I}_0)\\ \chi(\alpha)\ni y}} \Big(\big(1+\eta^2\delta_{x,y}\big)\delta_x(k_\alpha) + 2\big(\eta^2+\delta_{x,y}\big)\big[\delta_x(\ell^*_\alpha)\,\|\ell_\alpha\| + \|\ell^*_\alpha\|\,\delta_x(\ell^*_\alpha)\big]\Big).$$

We observe that both C_0 in (2.11) and M in (2.13) are finite whenever the family $\{k_\alpha, \ell_\alpha\}_{\alpha\in\mathcal{I}}$ is translation covariant and has finite range. For $\Lambda \in \mathcal{P}$ we denote by

$\mathcal{L}_\Lambda$ the bounded Lindblad generator on $\mathcal{A}$ defined by

$$\mathcal{L}_\Lambda := \sum_{X \subset \Lambda} L_X \tag{2.14}$$

and by $(\mathcal{P}_t^\Lambda)_{t\geq 0}$ the corresponding QMS. We finally recall that λ_1 is the spectral gap of the unperturbed one-qudit generator L_0.

Theorem 2.2. *Assume $C_0, M < \infty$ and consider $\mathcal{L}$ as an operator on $\mathcal{A}$ with domain $\mathcal{A}^1$. Then*

(i) the graph norm closure $\bar{\mathcal{L}}$ of $\mathcal{L}$ generates a QMS $(\mathcal{P}_t)_{t\geq 0}$ on $\mathcal{A}$;
(ii) $\mathcal{A}^1$ is a core for $\bar{\mathcal{L}}$;
(iii) for each $t \geq 0$ the operator $\mathcal{P}_t$ is the strong limit of $\mathcal{P}_t^\Lambda$ as $\Lambda \uparrow \mathbb{Z}^d$;
(iv) $|||\mathcal{P}_t f||| \leq e^{(M-\lambda_1)t} |||f|||$, for any $f \in \mathcal{A}^1$ and $t \geq 0$;
(v) the QMS $(\mathcal{P}_t)_{t\geq 0}$ has at least one stationary state.

We emphasize that in the above statement we can take $\mathcal{L}_0 = 0$, letting for example ρ be the normalized trace on H and $\{e_h\}_{h=0}^N$ be any orthonormal basis with respect to the normalized Hilbert-Schmidt inner product. In general, Theorem 2.2 provides sufficient conditions for the existence of the dynamics corresponding to a Lindblad generator with local jump operators. In the particular case in which $\mathcal{L} = i[\mathcal{K}, \cdot\,]$ is the Heisenberg operator associated with the (informal) infinite volume Hamiltonian $\mathcal{K} = \sum_{X\in\mathcal{P}} K_X$, with K_X as defined below (2.9), we deduce the existence of a one-parameter group of automorphisms $(\mathcal{U}_t)_{t\in\mathbb{R}}$ describing the Heisenberg evolution on $\mathcal{A}$. In this case, the assumptions of Theorem 2.2 are analogous to classical conditions in the literature, see e.g. [22, Thm. 7.6.2].

The next result, which provides a perturbative criterion for the ergodicity of the QMS $(\mathcal{P}_t)_{t\geq 0}$, depends instead on the non-vanishing of $\mathcal{L}_0$ and more precisely on the existence of a strictly positive spectral gap for the one-qudit dynamics, $\lambda_1 > 0$. If the family $\{k_\alpha, \ell_\alpha\}_{\alpha\in\mathcal{I}}$ has finite range the corresponding unique stationary state has exponentially decaying correlations.

Theorem 2.3. *Assume (2.11) and $M < \lambda_1$. Then*

(i) the QMS $(\mathcal{P}_t)_{t\geq 0}$ has a unique stationary state π;
(ii) for any $f \in \mathcal{A}^1$ and $t \geq 0$

$$\big\|\mathcal{P}_t f - \pi(f)\mathbf{1}\big\| \leq \frac{C_0}{\lambda_1 - M}\, e^{-(\lambda_1 - M)t}\, |||f|||;$$

(iii) if furthermore $\{k_\alpha, \ell_\alpha\}_{\alpha\in\mathcal{I}}$ has finite range then there exist $C, \zeta > 0$ such that for any $\Lambda_1, \Lambda_2 \in \mathcal{P}$ and any $f_1 \in \mathcal{A}_{\Lambda_1}$, $f_2 \in \mathcal{A}_{\Lambda_2}$

$$\big|\pi(f_1 f_2) - \pi(f_1)\pi(f_2)\big| \leq C e^{-\zeta\,\mathrm{dist}(\Lambda_1,\Lambda_2)} \big(\|f_1\| + |||f_1|||\big)\big(\|f_2\| + |||f_2|||\big).$$

The condition $M < \lambda_1$ can be explicitly checked for specific models, we refer to the Sections 4.1 and 4.3 for the cases of quantum spin systems and of the XYZ-model with site dissipation. In this respect, items (ii) and (iii) provide quantitative bounds on the speed of convergence to the stationary state and on the spatial decay of correlations for the stationary state.

By the density of $\mathcal{A}^1$ in $\mathcal{A}$, Theorem 2.3(ii) implies that for each $\mu \in \mathcal{S}$ the sequence $\mu\mathcal{P}_t$ converges weakly* to π. Even in the case in which $\mathcal{L} = \mathcal{L}_0$, it does not appear however possible to obtain a quantitative bound on this convergence uniformly in $\mu \in \mathcal{S}$. On the other hand, as we next discuss, if we restrict to translation covariant interactions and translation invariant $\mu \in \mathcal{S}$, there is a natural distance on the set of translation invariant states such that the distance between $\mu\mathcal{P}_t$ and π vanishes exponentially uniformly in μ. More precisely, the final topic that we discuss is the exponential convergence of the QMS $(\mathcal{P}_t)_{t\geq 0}$ to the unique stationary state π in term of the specific quantum one-Wasserstein distance introduced in [15], which is the non-commutative counterpart of the Ornstein $\bar{d}$ distance on the set of translation invariant probabilities. Given $\Lambda \in \mathcal{P}$ let $\|\cdot\|_{W_\Lambda}$ be the norm on the space $\mathcal{O}_\Lambda^0$ of self-adjoint and traceless elements in $\mathcal{A}_\Lambda$ defined by

$$\|\Delta\|_{W_\Lambda} := \frac{1}{2}\inf\Big\{\sum_{x\in\Lambda}\|\Delta^{(x)}\|_{\Lambda,\mathrm{Tr}}\colon \Delta^{(x)}\in\mathcal{O}_\Lambda^0,\ \mathrm{Tr}_{\{x\}}\Delta^{(x)}=0,\ \sum_{x\in\Lambda}\Delta^{(x)}=\Delta\Big\},$$

where $\|\cdot\|_{\Lambda,\mathrm{Tr}}$ is the trace norm on $\mathcal{A}_\Lambda$, i.e. $\|f\|_{\Lambda,\mathrm{Tr}} = \mathrm{Tr}(\sqrt{ff^*})$, and $\mathrm{Tr}_{\{x\}}\colon \mathcal{A}_\Lambda \to \mathcal{A}_{\Lambda\setminus\{x\}}$ denotes the partial trace on $\mathcal{H}_{\{x\}}$. Denoting by $\mathcal{S}_\Lambda$ the set of states on $\mathcal{A}_\Lambda$, the *quantum one-Wasserstein distance* W_Λ on $\mathcal{S}_\Lambda$ is defined by $W_\Lambda(\mu,\nu) := \|\mu-\nu\|_{W_\Lambda}$. Here we have identified $\mathcal{S}_\Lambda$ with the positive elements in $\mathcal{A}_\Lambda$ with unit trace. Recalling that $\mathcal{S}_\tau$ is the set of translation invariant states on $\mathcal{A}$, as proven in [15, Prop. 4.1], the *specific quantum one-Wasserstein distance* is the distance on $\mathcal{S}_\tau$ defined by

$$w(\mu,\nu) := \sup_{\Lambda\in\mathcal{P}}\frac{1}{|\Lambda|}W_\Lambda(\mu_\Lambda,\nu_\Lambda) = \lim_{\Lambda\uparrow\mathbb{Z}^d}\frac{1}{|\Lambda|}W_\Lambda(\mu_\Lambda,\nu_\Lambda), \tag{2.15}$$

where μ_Λ denotes the restriction of the state $\mu\in\mathcal{S}_\tau$ to a state on $\mathcal{A}_\Lambda$. Observe that the topology on $\mathcal{S}_\tau$ induced by w is finer than the weak* topology.

Theorem 2.4. *Assume* (2.11)*, $M < \lambda_1$, and that $\{k_\alpha,\ell_\alpha\}_{\alpha\in\mathcal{I}}$ is translation covariant. Then the unique stationary state π of the QMS $(\mathcal{P}_t)_{t\geq 0}$ is translation invariant and there exists a constant $C>0$ such that for any $\mu\in\mathcal{S}_\tau$ and $t\geq 0$*

$$w(\mu\mathcal{P}_t,\pi) \leq Ce^{-(\lambda_1-M)t}.$$

The proof of both Theorems 2.2 and 2.3 follows the strategy used in the construction of the Markov semigroup describing the evolution of interacting classical lattice systems [18, Ch. I]. The key ingredient is an a priori bound on the resolvent equation $(\lambda-\mathcal{L})g=f$ showing that $g\in\mathcal{A}^1$ whenever $f\in\mathcal{A}^1$. By approximating $\mathcal{L}$ with the finite volume generator $\mathcal{L}_\Lambda$ and using the Lumer-Phillips theorem, this a priori bound implies the existence of the infinite volume dynamics. As in the commutative case, when $M<\lambda_1$, the a priori bound obtained on the resolvent equation actually implies that the seminorm $|\!|\!|\cdot|\!|\!|$ is exponentially contracted by the QMS $(\mathcal{P}_t)_{t\geq 0}$. By routine arguments, this yields the exponential convergence to equilibrium stated in Theorem 2.3(ii). The exponential decay of spatial correlation at equilibrium in Theorem 2.3(iii) follows from Theorem 2.3(ii) and the "finite speed of propagation" of the QMS $(\mathcal{P}_t)_{t\geq 0}$, see [18, §I.4] for the corresponding statement in the commutative case. While Theorem 2.4 is a straightforward consequence of Theorem 2.3, its formulation appears novel also in the context of interacting classical lattice systems.

From a technical viewpoint, in the commutative case discussed in [18, Ch. I] the seminorm $|\!|\!|f|\!|\!|$ is defined in terms of the oscillations of f at the sites $x\in\mathbb{Z}^d$ while here it is adapted to the unperturbed dynamics. Correspondingly, while in [18, Ch. I] the strength of the unperturbed dynamics is specified by a Doeblin condition on the transition rates, here it is measured by the spectral gap λ_1 of the unperturbed one-qudit generator. Accordingly, a crucial input for the derivation of the a priori bound on the resolvent equation is the intertwining relationship (1.2) for the unperturbed generator.

3. Interacting qudits: proofs

In this section we prove Theorems 2.2, 2.3, and 2.4.

3.1. **Semigroup generation.**

Recalling the definition of η below (2.2) we first show that $\|E_{x,h}\|_{\mathcal{A}\to\mathcal{A}} \le \eta$.

Lemma 3.1. *For each $x \in \mathbb{Z}^d$ and $h \in \{0, 1, \dots, N\}$, the operator $E_{x,h}$ defined by (2.5) extends to a bounded operator on $\mathcal{A}$. In fact, for each $f \in \mathcal{A}$ we have $\|E_{x,0}f\| \le \|f\|$ and $\|E_{x,h}f\| \le \eta \|f\|$, $h \in \{1, \dots, N\}$.*

Proof. By the density of $\mathcal{A}^0$ in $\mathcal{A}$, it suffices to prove the stated inequalities for $f \in \mathcal{A}_\Lambda$ with $\Lambda \in \mathcal{P}$. We first observe that, by the very definition of $E_{x,h}$, we have $E_{x,h}f = E_{x,0}(e^*_{x,h}f)$. On the other hand, for $g \in \mathcal{A}_\Lambda$ we have $E_{x,0}g = \mathrm{Tr}_{\{x\}}(\rho g) \otimes \mathbb{1}_{\mathcal{H}_{\{x\}}}$ in which $\mathrm{Tr}_{\{x\}} \colon \mathcal{A}_\Lambda \to \mathcal{A}_{\Lambda\setminus\{x\}}$ is the partial trace on $H \simeq \mathcal{H}_{\{x\}}$. Since $g \mapsto \mathrm{Tr}_{\{x\}}(\rho g)$ is completely positive and unital, the Kadison-Schwarz inequality implies $\|E_{x,0}g\| \le \|g\|$. As $e_{x,0} = \mathbf{1}$ and $\|e_{x,h}\| \le \eta$, $h \in \{1, \dots, N\}$, this bound yields the claim. □

Proof of Lemma 2.1. The statement is a direct consequence of the following bound. For each $X \in \mathcal{P}$, $u \in \mathcal{A}_X$, and $f \in \mathcal{A}^1$,

$$\|[u, f]\| \le 2\eta\|u\| \sum_{x\in X} \delta_x(f). \tag{3.1}$$

To prove this inequality, given $x \in \mathbb{Z}^d$ define the operator $F_x \colon \mathcal{A} \to \mathcal{A}$ by

$$F_x f = \sum_{h=1}^{N} (E_{x,h}f)e_{x,h}, \tag{3.2}$$

so that (2.6) can be recast as $f = E_{x,0}f + F_x f$. Enumerating the elements of $X = \{x_1, \dots, x_m\}$ and using recursively this identity,

$$f = \Big(\prod_{j=1}^{m} E_{x_j,0}\Big) f + \sum_{j=1}^{m} \Big(\prod_{i<j} E_{x_i,0}\Big) F_{x_j} f. \tag{3.3}$$

Since the first term on the right-hand side commutes with $u \in \mathcal{A}_X$, the bound (3.1) follows by observing that $\|[f_1, f_2]\| \le 2\|f_1\|\|f_2\|$ and $\|E_{x,0}\|_{\mathcal{A}\to\mathcal{A}} = 1$, $\|F_x f\| \le \eta\delta_x(f)$, $x \in \mathbb{Z}^d$. □

In order to construct the QMS generated by the operator $\mathcal{L}$ defined in (2.9), we shall use the terminology and the results of [21, §X.8]. In particular, a densely defined operator T on a Banach space $\mathcal{X}$ with domain $\mathcal{D}$ is *accretive* if for each $x \in \mathcal{D}$ there exists $\wp \in \mathcal{X}'$ such that $\|\wp\|_{\mathcal{X}'} = 1$, $\wp(x) = \|x\|_{\mathcal{X}}$, and $\mathrm{Re}(\wp(Tx)) \ge 0$. By the finite dimensional theory of quantum Markov semigroups, the bounded operator L_X, $X \in \mathcal{P}$, as defined in (2.10), generates a QMS on the finite dimensional C^*-algebra $\mathcal{A}_X$. The Lumer-Philips theorem [21, Thm. X.48] thus implies that $-L_X$ is accretive and therefore also $-\mathcal{L}$, with domain $\mathcal{A}^1$, is accretive. For the sake of completeness, we however next provide a direct proof of the accretivity of $-\mathcal{L}$.

Lemma 3.2. *The operator $-\mathcal{L}$ with domain $\mathcal{A}^1$ is accretive.*

Proof. By the definition of $\mathcal{A}^1$, Lemma 2.1, and the Banach-Alaoglu theorem, it is enough to show that for each $f \in \mathcal{A}_\Lambda$, with $\Lambda \in \mathcal{P}$, there exists $\wp \in \mathcal{A}'$ such that

$$\|\wp\|_{\mathcal{A}'} = 1, \qquad \wp(f) = \|f\|, \qquad \mathrm{Re}(\wp(\mathcal{L}f)) \leq 0. \tag{3.4}$$

Since $\mathcal{A}_\Lambda$ is finite dimensional, there exists $\xi \in \mathcal{H}$ with $\|\xi\|_{\mathcal{H}} = 1$, which is eigenvector of ff^* with maximal eigenvalue: $(ff^*)\xi = \|f\|^2\xi$. Let $\wp \in \mathcal{A}'$ be defined by

$$\wp(g) = \frac{(\xi, gf^*\xi)_{\mathcal{H}}}{\|f\|}$$

where $(\cdot,\cdot)_{\mathcal{H}}$ denotes the inner product in $\mathcal{H}$. We claim that this functional fulfil the three conditions in (3.4). The second condition holds trivially. The first one follows from the second and the bound

$$|\wp(g)| \leq \frac{\|gf^*\xi\|_{\mathcal{H}}}{\|f\|} \leq \frac{\|gf^*\|}{\|f\|} \leq \frac{\|g\|\|f^*\|}{\|f\|} = \|g\|.$$

Recalling (2.9), in order to prove the third condition in (3.4) it suffices to show that for each self-adjoint $k \in \mathcal{A}$ and $u \in \mathcal{A}$

$$\mathrm{Re}\,\wp\big(i[k,f] + [u^*,f]u + u^*[f,u]\big) \leq 0. \tag{3.5}$$

For the first term we have

$$\mathrm{Re}\,\wp(i[k,f]) = -\frac{1}{\|f\|}\,\mathrm{Im}(\xi,[k,f]f^*\xi)_{\mathcal{H}} = \frac{1}{\|f\|}\,\mathrm{Im}\,\big((f^*\xi,kf^*\xi)_{\mathcal{H}} - \|f\|^2(\xi,k\xi)_{\mathcal{H}}\big) = 0$$

since k is self-adjoint. On the other hand,

$$\begin{aligned}
\mathrm{Re}\,\wp([u^*,f]u + u^*[f,u]) &= \mathrm{Re}\,\wp(2u^*fu - u^*uf - fu^*u) \\
&= \frac{1}{\|f\|}\,\mathrm{Re}\,\big(\xi,(2u^*fu - u^*uf - fu^*u)f^*\xi\big)_{\mathcal{H}} \\
&= \frac{1}{\|f\|}\,\mathrm{Re}\,\Big(2(\xi,u^*fuf^*\xi)_{\mathcal{H}} - (\xi,u^*uff^*\xi)_{\mathcal{H}} - (\xi,fu^*uf^*\xi)_{\mathcal{H}}\Big) \\
&= \frac{1}{\|f\|}\,\mathrm{Re}\,\Big(2(f^*u\xi,uf^*\xi)_{\mathcal{H}} - \|f\|^2\|u\xi\|^2_{\mathcal{H}} - \|uf^*\xi\|^2_{\mathcal{H}}\Big) \\
&\leq \frac{1}{\|f\|}\big(\|f^*u\xi\|^2_{\mathcal{H}} - \|f\|^2\|u\xi\|^2_{\mathcal{H}}\big) \leq 0,
\end{aligned}$$

where we used Cauchy-Schwarz in the second last step. The proof of (3.5) is thus completed. □

We next prove the intertwining relationship between the unperturbed generator $\mathcal{L}_0$ and the operators $E_{x,h}$ defined by (2.5).

Lemma 3.3. *For each $f \in \mathcal{A}^1$, $x \in \mathbb{Z}^d$, and $h \in \{0,1,\dots,N\}$,*

$$E_{x,h}\mathcal{L}_0 f - \mathcal{L}_0 E_{x,h} f = -\lambda_h E_{x,h} f.$$

Proof. By linearity and density it is enough to prove the statement for a monomial, $f = \otimes_y f_y$. Recalling (2.4), from the spectral decomposition (2.2) and the definition

(2.5) of $E_{x,h}$ we deduce

$$\begin{aligned} E_{x,h}\mathcal{L}_0 f &= -\lambda_h \langle e_h, f_x\rangle_\rho \big(\otimes_{y\neq x} f_y\big) \\ &\qquad - \sum_{z\neq x}\sum_{k=1}^{N} \lambda_k \langle e_k, f_z\rangle_\rho \langle e_h, f_x\rangle_\rho \big(\otimes_{y\neq x,z} f_y\big)\otimes e_{z,k}, \\ \mathcal{L}_0 E_{x,h} f &= - \sum_{z\neq x}\sum_{k=1}^{N} \lambda_k \langle e_k, f_z\rangle_\rho \langle e_h, f_x\rangle_\rho \big(\otimes_{y\neq x,z} f_y\big)\otimes e_{z,k}. \end{aligned}$$

The statement follows. □

The following lemma provides the key estimate in realizing the generator $\mathcal{L}$ as a perturbation of $\mathcal{L}_0$. Recall that $\mathcal{L}_1 = \sum_{\alpha\in\mathcal{I}} L^1_\alpha$ with L^1_α defined in (2.12) and that $\delta(f)\in\ell_1(\mathbb{Z}^d)$ has been defined in (2.8).

Lemma 3.4. *For each $\alpha\in\mathcal{I}$, $f\in\mathcal{A}^1$, $x\in\mathbb{Z}^d$, and $h\in\{1,\dots,N\}$,*

$$\big\| E_{x,h}L^1_\alpha f - L^1_\alpha E_{x,h} f\big\| \le \sum_{y\in\mathbb{Z}^d} \theta_{x,y}(\alpha)\delta_y(f)$$

where

$$\begin{aligned} &\theta_{x,y}(\alpha) := 2\eta(1+\eta^2\delta_{x,y})\delta_x(k_\alpha) \\ &+4\eta^3(1+\eta^{-2}\delta_{x,y})\times \begin{cases} \delta_x\big(\ell^{0\,*}_{\alpha_0}\big)\big\|\ell^1_{\alpha_0}\big\| + \big\|\ell^{0\,*}_{\alpha_0}\big\|\,\delta_x\big(\ell^1_{\alpha_0}\big) \\ \quad + \delta_x\big(\ell^{1\,*}_{\alpha_0}\big)\big\|\ell^0_{\alpha_0}\big\| + \big\|\ell^{1\,*}_{\alpha_0}\big\|\,\delta_x\big(\ell^0_{\alpha_0}\big) & \text{if } \alpha = \imath(\alpha_0)\in\imath(\mathcal{I}_0). \\ \quad + \delta_x\big(\ell^{1\,*}_{\alpha_0}\big)\big\|\ell^1_{\alpha_0}\big\| + \big\|\ell^{1\,*}_{\alpha_0}\big\|\,\delta_x\big(\ell^1_{\alpha_0}\big) \\ \\ \delta_x(\ell^*_\alpha)\,\|\ell_\alpha\| + \|\ell^*_\alpha\|\,\delta_x(\ell^*_\alpha) & \text{if } \alpha\notin\imath(\mathcal{I}_0) \end{cases} \end{aligned}$$

Proof. The statement is a direct consequence of the bounds (3.6) and (3.7) below.

For each $u\in\mathcal{A}^0$, $x\in\mathbb{Z}^d$, $h\in\{1,\dots,N\}$, and $f\in\mathcal{A}^1$,

$$\big\|E_{x,h}[u,f] - [u, E_{x,h}f]\big\| \le \sum_{y\in\mathbb{Z}^d}\gamma_{x,y}(u)\delta_y(f), \tag{3.6}$$

where

$$\gamma_{x,y}(u) = 2\eta(1+\eta^2\delta_{x,y})\delta_x(u).$$

For each $u,v\in\mathcal{A}^0$, $x\in\mathbb{Z}^d$, $h\in\{1,\dots,N\}$, and $f\in\mathcal{A}^1$,

$$\big\|E_{x,h}\big(u[f,v]+[u,f]v\big) - u[E_{x,h}f,v] - [u,E_{x,h}f]v\big\| \le \sum_{y\in\mathbb{Z}^d}\gamma_{x,y}(u,v)\delta_y(f), \tag{3.7}$$

where

$$\gamma_{x,y}(u,v) = 4\eta^3\big(1+\eta^{-2}\delta_{x,y}\big)\Big(\delta_x(u)\|v\| + \|u\|\delta_x(v)\Big).$$

To prove (3.6), let $X=\{x_1,\dots,x_m\}\in\mathcal{P}$ with $x_1=x$ and $u\in\mathcal{A}_X$. By (2.6) and (3.2)

$$\begin{aligned} &E_{x,h}[u,f] - [u,E_{x,h}f] = E_{x,h}[u,E_{x,0}f] \\ &\qquad + \sum_{k=1}^{N}\Big\{E_{x,h}(ue_{x,k})(E_{x,k}f) - (E_{x,k}f)E_{x,h}(e_{x,k}u)\Big\} - \big[u,E_{x,h}f\big] \end{aligned} \tag{3.8}$$

where we used that $E_{x,h} e_{x,k} = \delta_{h,k}$. By (3.3) and recalling that $e_0 = 1\!\!1_H$, the first term on the right-hand side above can be expanded as

$$E_{x,h}[u, E_{x,0} f] = \big[E_{x_1,h} u, E_{x_1,0} f\big] = \sum_{i=2}^{m} \Big[E_{x_1,h} u \,, \Big(\prod_{j<i} E_{x_j,0}\Big) F_{x_i} f\Big]$$

whose operator norm is bounded above by

$$2\|E_{x,h} u\| \sum_{y \in X \setminus \{x\}} \|F_y f\| \le 2\eta \|E_{x,h} u\| \sum_{y \in X \setminus \{x\}} \sum_{k=1}^{N} \|E_{y,k} f\|.$$

On the other hand, by using again the identity (2.6) for u, the second term on the right-hand side of (3.8) can be rewritten as

$$\big[E_{x,0} u, E_{x,h} f\big] + \sum_{k,j=1}^{N} \Big\{ C^h_{j,k} \big(E_{x,j} u\big)\big(E_{x,k} f\big) - C^h_{k,j} \big(E_{x,k} f\big)\big(E_{x,j} u\big) \Big\} \tag{3.9}$$

where $C^h_{j,k} := E_{x,h}(e_{x,j} e_{x,k})$. Since $C^h_{j,k} = c^h_{j,k} \mathbf{1}$ with $|c^h_{j,k}| \le \eta^3$, the operator norm of the second term in (3.9) is bounded above by

$$2\eta^3 \sum_{j=1}^{N} \|E_{x,j} u\| \sum_{k=1}^{N} \|E_{x,k} f\|.$$

Finally, again by the identity (2.6) for u, the sum of the third term on the right-hand side of (3.8) and the first term of (3.9) gives

$$-\sum_{k=1}^{N} \big[(E_{x,k} u) e_{x,k}, E_{x,h} f\big]$$

whose operator norm is bounded by

$$2\eta \sum_{k=1}^{N} \|E_{x,k} u\| \, \|E_{x,h} f\|.$$

Gathering the previous estimates we deduce (3.6).

To prove (3.7), let $X \in \mathcal{P}$ be such that $X \ni x$ and $u, v \in \mathcal{A}_X$. Enumerate as before the elements in $X \in \mathcal{P}$ letting $x_1 = x$ so that $X = \{x_1, \dots, x_m\}$. By the identity (2.6),

$$\begin{aligned} & E_{x,h}\big(u[f,v] + [u,f]v\big) - u[E_{x,h} f, v] - [u, E_{x,h} f]v\big) \\ & = E_{x,h}\big(u[E_{x,0} f, v] + [u, E_{x,0} f]v + 2u(F_x f)v - uv(F_x f) - (F_x f)uv\big) \\ & \quad - 2u(E_{x,h} f)v + uv(E_{x,h} f) + (E_{x,h} f)uv. \end{aligned} \tag{3.10}$$

By (3.3)

$$\begin{aligned} E_{x,h}\big(u[E_{x,0} f, v] + [u, E_{x,0} f]v\big) = \sum_{i=2}^{m} E_{x,h} \Big\{ 2u\Big(\Big(\prod_{j<i} E_{x_j,0}\Big) F_{x_i} f\Big) v \\ - uv\Big(\Big(\prod_{j<i} E_{x_j,0}\Big) F_{x_i} f\Big) - \Big(\Big(\prod_{j<i} E_{x_j,0}\Big) F_{x_i} f\Big) uv \Big\}. \end{aligned} \tag{3.11}$$

We claim that for $g \in \mathcal{A}_{\{x\}^c}$ and $u, v \in \mathcal{A}$

$$\big\|E_{x,h}\big(ugv\big)\big\| \le \eta^2 \|g\| \big(\delta_x(u) \|v\| + \|u\| \delta_x(v)\big). \tag{3.12}$$

Indeed, by using the identity (2.6) for u,

$$E_{x,h}\big(ugv\big) = (E_{x,0}u)g(E_{x,h}v) + \sum_{j=1}^{N}(E_{x,j}u)\, E_{x,h}(e_{x,j}gv)$$

so that

$$\big\|E_{x,h}\big(ugv\big)\big\| \le \|g\|\Big(\|u\|\|E_{x,h}v\| + \eta^2 \sum_{j=1}^{N} \|E_{x,j}u\|\|v\|\Big)$$

which implies (3.12). By using (3.12) we then bound the operator norm of the right-hand side of (3.11) by

$$4\eta^3 \sum_{j=1}^{N} \big(\|E_{x,j}u\|\|v\| + \|u\|\|E_{x,j}v\|\big) \sum_{y\in X\setminus\{x\}} \sum_{k=1}^{N} \big\|E_{y,k}f\big\|.$$

By using (2.6) and (3.3) first for u and then for v, noticing that $E_{x,h}F_x = E_{x,h}$,

$$\begin{aligned} E_{x,h}\big(2u(F_xf)v - uv(F_xf) - (F_xf)uv\big) &= 2(E_{x,0}u)(E_{x,h}f)(E_{x,0}v) \\ &- (E_{x,0}u)(E_{x,0}v)(E_{x,h}f) - (E_{x,h}f)(E_{x,0}u)(E_{x,0}v) \\ &+ \sum_{k,j=1}^{N} \Big(2C^h_{k,j}(E_{x,0}u)(E_{x,k}f)(E_{x,j}v) - C^h_{j,k}(E_{x,0}u)(E_{x,j}v)(E_{x,k}f) \\ &\qquad - C^h_{k,j}(E_{x,k}f)(E_{x,0}u)(E_{x,j}v)\Big) \\ &+ E_{x,h}\Big(2(F_xu)(F_xf)v - (F_xu)v(F_xf) - (F_xf)(F_xu)v\Big), \end{aligned} \tag{3.13}$$

where $C^h_{k,j}$ is defined below (3.9). Using that $\|C^h_{k,j}\| \le \eta^3$, the operator norm of the third and fourth lines on (3.13) is bounded by

$$4\eta^3\, \|u\| \sum_{j=1}^{N} \|E_{x,j}v\| \sum_{k=1}^{N} \|E_{x,k}f\|.$$

By using Lemma 3.1, the operator norm of the fifth line on (3.13) can be bounded by

$$4\eta^3\, \|v\| \sum_{j=1}^{N} \|E_{x,j}u\| \sum_{k=1}^{N} \|E_{x,k}f\|.$$

Finally, again by the identity (2.6) for u and v, the sum of the last line on the right-hand side of (3.10) and the first three terms on the right-hand side of (3.13) gives

$$\begin{aligned} &- 2(E_{x,0}u)(E_{x,h}f)(F_xv) + (E_{x,0}u)(F_xv)(E_{x,h}f) + (E_{x,h}f)(E_{x,0}u)(F_xv) \\ &- 2(F_xu)(E_{x,h}f)v + (F_xu)v(E_{x,h}f) + (E_{x,h}f)(F_xu)v \end{aligned}$$

whose operator norm is bounded by

$$4\eta \sum_{j=1}^{N} \big(\|E_{x,j}u\|\|v\| + \|u\|\|E_{x,j}v\|\big)\, \|E_{x,h}f\|.$$

Gathering the previous estimates we deduce (3.7). □

Recalling $\theta_{x,y}$ has been defined below (2.13), let Θ be the operator on $\ell_1(\mathbb{Z}^d)$ with kernel $\big(\theta_{x,y}\big)_{x,y\in\mathbb{Z}^d}$. Namely,

$$(\Theta\beta)_x = \sum_{y\in\mathbb{Z}^d} \theta_{x,y}\beta_y, \qquad x\in\mathbb{Z}^d,\ \beta\in\ell_1(\mathbb{Z}^d). \tag{3.14}$$

Recalling that λ_1 is the spectral gap of the unperturbed single site Lindblad generator, the following lemma provides an a priori bound on the resolvent equation.

Lemma 3.5. *Assume* (2.11) *and* $M < +\infty$. *The operator* Θ *on* $\ell_1(\mathbb{Z}^d)$ *satisfies* $\|\Theta\|_{\ell_1\to\ell_1} \le M$. *Furthermore, if* $f, g \in \mathcal{A}^1$ *satisfy*

$$(\lambda - \mathcal{L})g = f \tag{3.15}$$

for some $\lambda > 0$ *such that* $\lambda + \lambda_1 > M$, *then*

$$\delta(g) \le \big(\lambda + \lambda_1 - \Theta\big)^{-1}\delta(f) \qquad \textit{pointwise.} \tag{3.16}$$

In particular,

$$|||g||| \le (\lambda + \lambda_1 - M)^{-1}|||f|||\,. \tag{3.17}$$

Proof. Definition (2.13) readily implies the bound $\|\Theta\|_{\ell_1\to\ell_1} \le M$. Assuming (3.15), for $x\in\mathbb{Z}^d$ and $h\in\{1,\dots,N\}$ we have

$$\begin{aligned}\lambda E_{x,h}g &= E_{x,h}f + E_{x,h}\mathcal{L}g\\ &= E_{x,h}f + \mathcal{L}E_{x,h}g + E_{x,h}\mathcal{L}_0 g - \mathcal{L}_0E_{x,h}g + E_{x,h}\mathcal{L}_1 g - \mathcal{L}_1E_{x,h}g.\end{aligned}$$

Applying Lemma 3.3, we thus get

$$(\lambda+\lambda_h)E_{x,h}g = E_{x,h}f + \mathcal{L}E_{x,h}g + E_{x,h}\mathcal{L}_1 g - \mathcal{L}_1E_{x,h}g\,. \tag{3.18}$$

Since, as proven in Lemma 3.2, $-\mathcal{L}$ is accretive, for each $x\in\mathbb{Z}^d$ and $h = 1,\dots,N$, there exists $\wp\in\mathcal{A}'$ such that $\|\wp\|_{\mathcal{A}'} = 1$, $\wp(E_{x,h}g) = \|E_{x,h}g\|$, and $\operatorname{Re}\wp(\mathcal{L}E_{x,h}g) \le 0$. Pairing both sides of (3.18) with $\wp$ and taking real parts we deduce

$$\begin{aligned}(\lambda+\lambda_h)\|E_{x,h}g\| &\le \operatorname{Re}\wp(E_{x,h}f) + \operatorname{Re}\wp(E_{x,h}\mathcal{L}_1 g - \mathcal{L}_1E_{x,h}g)\\ &\le \|E_{x,h}f\| + \|E_{x,h}\mathcal{L}_1 g - \mathcal{L}_1E_{x,h}g\|.\end{aligned} \tag{3.19}$$

Lemma 3.4 and the definition of $\theta_{x,y}$ below (2.13) imply that for each $h\in\{1,\dots,N\}$

$$\|E_{x,h}\mathcal{L}_1 g - \mathcal{L}_1E_{x,h}g\| \le \frac{1}{N}\sum_{y\in\mathbb{Z}^d}\theta_{x,y}\delta_y(g).$$

Since $\lambda_h \ge \lambda_1$, summing over $h\in\{1,\dots,N\}$ the bound (3.19) we get

$$(\lambda+\lambda_1)\delta(g) \le \delta(f) + \Theta\delta(g) \qquad \text{pointwise.}$$

Equivalently,

$$\delta(g) \le \frac{1}{\lambda+\lambda_1}\delta(f) + \frac{1}{\lambda+\lambda_1}\Theta\delta(g) \qquad \text{pointwise.}$$

Since Θ is positive operator and $\delta(f), \delta(g)\in\ell_1(\mathbb{Z}^d)$, this inequality can be iterated to obtain

$$\delta(g) \le \sum_{k=0}^{n-1}\frac{\Theta^k}{(\lambda+\lambda_1)^{k+1}}\delta(f) + \frac{\Theta^n}{(\lambda+\lambda_1)^n}\delta(g) \qquad \text{pointwise.}$$

By assumption $\lambda+\lambda_1 > M$, hence the last term in the right-hand side above vanishes as $n\to\infty$. The bound (3.16) follows. Since $\|(\lambda+\lambda_1-\Theta)^{-1}\|_{\ell_1\to\ell_1} \le (\lambda+\lambda_1-M)^{-1}$, the bound (3.17) is obtained by taking the $\ell_1(\mathbb{Z}^d)$-norm of both sides in (3.16). □

Proof of Theorem 2.2. By Lemma 3.2 $-\mathcal{L}$ is accretive thus closable, see e.g. [21, Ex. X.52]. The proof that its closure $\bar{\mathcal{L}}$ generates a QMS on $\mathcal{A}$ is achieved by the following steps.

Step 1. The operator $\bar{\mathcal{L}}$ generates a strongly continuous contraction semigroup $(\mathcal{P}_t)_{t\geq 0}$ on the Banach space $\mathcal{A}$.

In view of the accretivity of $-\mathcal{L}$ and the Lumer-Phillips theorem, see e.g. [21, Thm. X.48], it is enough to show $\mathrm{Ran}(\lambda - \bar{\mathcal{L}}) = \mathcal{A}$ for some $\lambda > 0$, where $\mathrm{Ran}(T)$ denotes the image of the linear operator T. To this end it suffices to show that $\mathrm{Ran}(\lambda - \mathcal{L})$ is dense in $\mathcal{A}$. Pick a sequence $\Lambda_n \in \mathcal{P}$, $n \in \mathbb{N}$, such that $\Lambda_n \subset \Lambda_{n+1}$ and $\bigcup_n \Lambda_n = \mathbb{Z}^d$. Let also $\mathcal{L}^{(n)}$ be the finite volume generator defined in (2.14) with Λ replaced by Λ_n and observe that $-\mathcal{L}^{(n)}$ is an accretive bounded operator on $\mathcal{A}$. Hence, $\mathcal{L}^{(n)}$ generates a strongly continuous contraction semigroup on $\mathcal{A}$ denoted by $\big(\mathcal{P}_t^{(n)}\big)_{t\geq 0}$. By the Lumer-Phillips theorem we then deduce $\mathrm{Ran}(\lambda - \mathcal{L}^{(n)}) = \mathcal{A}$ for any $n \geq 1$ and $\lambda > 0$. Fix $f \in \mathcal{A}^1$, choose $\lambda > 0$ such that $\lambda + \lambda_1 > M$, and set $g_n = (\lambda - \mathcal{L}^{(n)})^{-1} f$, $f_n = (\lambda - \mathcal{L}) g_n$. We claim that

$$\|f_n - f\| \to 0 \quad \text{as} \quad n \to \infty\,. \tag{3.20}$$

Since $\mathcal{A}^1$ is dense in $\mathcal{A}$, this yields the density of $\mathrm{Ran}(\lambda - \mathcal{L})$ in $\mathcal{A}$.

To prove (3.20) we decompose $\mathcal{L}^{(n)} = \mathcal{L}_0^{(n)} + \mathcal{L}_1^{(n)}$ where $\mathcal{L}_0^{(n)} = \sum_{x\in\Lambda_n} L_x^0$ and $\mathcal{L}_1^{(n)} = \sum_{\alpha:\, \chi(\alpha)\subset\Lambda_n} L_\alpha^1$. Set $\theta_{x,y}^{(n)} = \theta_{x,y}$ if $x, y \in \Lambda_n$ and $\theta_{x,y}^{(n)} = 0$ otherwise and let $\Theta^{(n)}$ be the operator on $\ell_1(\mathbb{Z}^d)$ with kernel $\theta_{x,y}^{(n)}$. We the claim that

$$\delta(g_n) \leq \big(\lambda + \lambda_1 - \Theta^{(n)}\big)^{-1} \delta(f_n), \qquad \text{pointwise.}$$

This follows indeed from Lemma 3.4 and the argument in the proof of Lemma 3.5 to deduce (3.16). Since $\theta_{x,y}^{(n)} \leq \theta_{x,y}$, the previous bound implies

$$\delta(g_n) \leq (\lambda + \lambda_1 - \Theta)^{-1} \delta(f), \qquad \text{pointwise.} \tag{3.21}$$

The argument in proof of Lemma 2.1 and more precisely the bound (3.1) implies

$$\|f_n - f\| = \big\|(\mathcal{L}^{(n)} - \mathcal{L}) g_n\big\| \leq \sum_{x\in\mathbb{Z}^d} C_0^{(n)}(x) \delta_x(g_n)$$

where

$$C_0^{(n)}(x) = 2\eta \sum_{\substack{\alpha\in\chi^{-1}(\{x\}) \\ \chi(\alpha)\cap\Lambda_n\neq\emptyset}} (\|k_\alpha\| + 2\|\ell_\alpha\|^2).$$

Recalling (2.11) we now observe that $C_0^{(n)}(x) \leq C_0$ and $C_0^{(n)}$ vanishes pointwise as $n \to \infty$. Hence, by the bound (3.21) and dominated convergence we conclude the proof of (3.20).

Step 2. $\mathcal{P}_t(\mathcal{A}^1) \subset \mathcal{A}^1$ and $|||\mathcal{P}_t f||| \leq e^{(M-\lambda_1)t} |||f|||$, $t \geq 0$ and $f \in \mathcal{A}^1$.

We first observe that, by the relationship between the semigroup $\mathcal{P}_t$ and the resolvent $(\lambda - \bar{\mathcal{L}})^{-1}$, for each $u \in \mathcal{A}$ and $t > 0$ we have

$$\mathcal{P}_t u = \lim_{n\to\infty} \Big(\frac{n}{t}\Big)^n \Big(\frac{n}{t} - \bar{\mathcal{L}}\Big)^{-n} u. \tag{3.22}$$

For f as in the statement, let g_n and f_n be defined as in Step 1, where we recall that $\lambda + \lambda_1 > M$. Since, as proven in Step 1, $f_n \to f$, and $(\lambda - \mathcal{L})^{-1}$ is bounded, $g_n = (\lambda - \mathcal{L})^{-1} f_n$ is a Cauchy sequence, whose limit is denoted by $g \in \mathcal{A}$. We deduce that the sequence $\mathcal{L} g_n = \lambda g_n - f_n$ has a limit. Hence, f belongs to the domain of $\bar{\mathcal{L}}$ and $f = (\lambda - \bar{\mathcal{L}})g$. By taking the limit $n \to \infty$ in (3.21), we deduce that $\delta(g) \le (\lambda + \lambda_1 - \Theta)\delta(f)$ pointwise, in particular $g \in \mathcal{A}^1$. Thus $(\lambda - \bar{\mathcal{L}})^{-1}\mathcal{A}^1 \subset \mathcal{A}^1$, which, by (3.22), proves that $\mathcal{P}_t \mathcal{A}^1 \subset \mathcal{A}^1$, $t \ge 0$.

As just proven, if $\lambda + \lambda_1 > M$,

$$\delta((\lambda - \bar{\mathcal{L}})^{-1} f) \le (\lambda + \lambda_1 - \Theta)^{-1}\delta(f) \qquad \text{pointwise.}$$

Since Θ is a positive operator, by iterating this bound we deduce that for every $n \in \mathbb{N}$

$$\delta((\lambda - \bar{\mathcal{L}})^{-n} f) \le (\lambda + \lambda_1 - \Theta)^{-n}\delta(f) \qquad \text{pointwise.}$$

Hence, by (3.22), for $t > 0$

$$\begin{aligned}
|||\mathcal{P}_t f||| &= \sum_{x\in\mathbb{Z}^d} \delta_x(\mathcal{P}_t f) = \sum_{x\in\mathbb{Z}^d} \lim_{n\to\infty} \Big(\frac{n}{t}\Big)^n \delta_x\Big(\Big(\frac{n}{t} - \bar{\mathcal{L}}\Big)^{-n} f\Big) \\
&\le \sum_{x\in\mathbb{Z}^d} \lim_{n\to\infty} \Big(\frac{n}{t}\Big)^n \Big(\Big(\frac{n}{t} + \lambda_1 - \Theta\Big)^{-n} \delta(f)\Big)_x \\
&= \sum_{x\in\mathbb{Z}^d} \big(e^{(\Theta-\lambda_1)t}\delta(f)\big)_x \le e^{(M-\lambda_1)t} |||f||| \,,
\end{aligned}$$

where we used $\|\Theta\|_{\ell_1\to\ell_1} \le M$ in the last inequality.

Step 3. For each $f \in \mathcal{A}$ and $t \ge 0$ the sequence $\mathcal{P}_t^{(n)} f$ converges to $\mathcal{P}_t f$ as $n \to \infty$.

Since $\mathcal{P}_t$ and $\mathcal{P}_t^{(n)}$ are contractions on $\mathcal{A}$, and $\mathcal{A}^1$ is dense in $\mathcal{A}$, it is enough to show the statement for each $f \in \mathcal{A}^1$. By Step 2 and standard interpolation, we have

$$\mathcal{P}_t f - \mathcal{P}_t^{(n)} f = \int_0^t ds \frac{d}{ds}\big(\mathcal{P}_{t-s}^{(n)} \mathcal{P}_s f\big) = \int_0^t ds\, \mathcal{P}_{t-s}^{(n)}(-\mathcal{L}^{(n)} + \mathcal{L})\mathcal{P}_s f \,,$$

so that

$$\|\mathcal{P}_t f - \mathcal{P}_t^{(n)} f\| \le \int_0^t ds \|(-\mathcal{L}^{(n)} + \mathcal{L})\mathcal{P}_s f\| \,.$$

By the argument below (3.21), dominated convergence, and again Step 2, we then conclude that the right-hand side vanishes as $n \to \infty$.

Conclusion. By Step 1, $\bar{\mathcal{L}}$ generates a strongly continuous contraction semigroup $(\mathcal{P}_t)_{t\ge0}$ on the Banach space $\mathcal{A}$. To show that $(\mathcal{P}_t)_{t\ge0}$ is a QMS we need to prove that $\mathcal{P}_t$ is a completely positive operator on the C^*-algebra $\mathcal{A}$ for each $t \ge 0$. Observing that $\mathcal{L}^{(n)}$ is a finite rank operator, the standard theory of QMS on finite dimensional C^*-algebra [19] implies that $\mathcal{P}_t^{(n)}$ is a completely positive operator on $\mathcal{A}$ for each $t \ge 0$ and $n \in \mathbb{N}$. As follows from Step 3, for each $t \ge 0$, the operator $\mathcal{P}_t$ is the strong limit of $\mathcal{P}_t^{(n)}$, and therefore also completely positive.

In view of Step 2 and [21, Thm. X.49] $\mathcal{A}^1$ is a core for $\bar{\mathcal{L}}$. Claims (iii) and (iv) are the content of Steps 2 and 3, respectively.

Finally, to prove item (v), pick $\mu \in \mathcal{S}$ and set

$$\mu^T := \frac{1}{T}\int_0^T \mu\mathcal{P}_t\, dt, \qquad T \in (0,\infty).$$

Since $\|\mu^T\|_{\mathcal{A}'} = 1$, the Banach-Alaoglu theorem yields the existence of $\pi \in \mathcal{A}'$ with $\|\pi\|_{\mathcal{A}'} \le 1$ and a sequence $T_n \to \infty$ such that $\mu^{T_n} \to \pi$ weakly* in $\mathcal{A}'$. Moreover, π

is positive and, since $\mathcal{A}$ is unital, $\pi(\mathbf{1}) = 1$ so that $\pi \in \mathcal{S}$. Fix $s \geq 0$, by taking the limit $n \to \infty$ in

$$\mu^{T_n} \mathcal{P}_s = \frac{1}{T_n} \int_0^{T_n} \mu \mathcal{P}_{t+s}\, dt = \mu^{T_n} - \frac{1}{T_n} \int_0^s \mu \mathcal{P}_t\, dt + \frac{1}{T_n} \int_{T_n}^{T_n+s} \mu \mathcal{P}_t\, dt,$$

we deduce that $\pi = \pi \mathcal{P}_s$ hence π is stationary. □

Remark 3.6. The proof of Theorem 2.2 given above actually implies that for each $f \in \mathcal{A}^1$ and $t \geq 0$

$$\delta(\mathcal{P}_t f) \leq e^{-\lambda_1 t} e^{t\Theta} \delta(f) \qquad \text{pointwise.}$$

3.2. **Perturbative criterion for ergodicity.** Before discussing the proof of Theorem 2.3, we show that the QMS $\big(\mathcal{P}_t\big)_{t \geq 0}$ has finite speed of propagation: the evolution of the product of two observables, with distant "support", can be approximated by the product of their respective evolution on a suitable time scale.

Proposition 3.7. *Set*

$$\omega_{x,y} := 8\eta^2 \sum_{\substack{\alpha \in \mathcal{I} \\ \chi(\alpha) \supset \{x,y\}}} \|\ell_\alpha\|^2, \qquad x, y \in \mathbb{Z}^d \tag{3.23}$$

and assume there exists $\xi > 0$ *such that*

$$M_\xi := \sup_{y \in \mathbb{Z}^d} \sum_{x \in \mathbb{Z}^d} \theta_{x,y} e^{\xi |x-y|} < \infty, \tag{3.24}$$

$$\Omega_\xi := \sup_{x,y \in \mathbb{Z}^d} \omega_{x,y} e^{\xi |x-y|} < \infty. \tag{3.25}$$

Then, for each $\Lambda_1, \Lambda_2 \in \mathcal{P}$, $f_1 \in \mathcal{A}_{\Lambda_1}$, $f_2 \in \mathcal{A}_{\Lambda_2}$, *and* $t \geq 0$,

$$\|\mathcal{P}_t(f_1 f_2) - (\mathcal{P}_t f_1)(\mathcal{P}_t f_2)\| \leq \Omega_\xi \frac{e^{2(M_\xi - \lambda_1)t} - 1}{2(M_\xi - \lambda_1)} e^{-\xi\, \mathrm{dist}(\Lambda_1, \Lambda_2)} |||f_1||| \, |||f_2|||.$$

Lemma 3.8. *For any* $f_1, f_2 \in \mathcal{A}^1$ *and* $t \geq 0$,

$$\|\mathcal{P}_t(f_1 f_2) - (\mathcal{P}_t f_1)(\mathcal{P}_t f_2)\| \leq \sum_{x,y \in \mathbb{Z}^d} \omega_{x,y} \int_0^t ds e^{-2\lambda_1 s} (e^{s\Theta} \delta(f_1))_x (e^{s\Theta} \delta(f_2))_y.$$

Proof. Set $F_t = \mathcal{P}_t(f_1 f_2) - (\mathcal{P}_t f_1)(\mathcal{P}_t f_2)$. By direct computation,

$$\frac{d}{dt} F_t = \mathcal{L} F_t + G_t$$

where

$$G_t := \mathcal{L}((\mathcal{P}_t f_1)(\mathcal{P}_t f_2)) - (\mathcal{L}\mathcal{P}_t f_1)(\mathcal{P}_t f_2) - (\mathcal{P}_t f_1)(\mathcal{L}\mathcal{P}_t f_2)\,.$$

Since $F_0 = 0$ and $\mathcal{P}_t$ is a contraction on $\mathcal{A}$ we deduce

$$\|F_t\| = \Big\| \int_0^t \mathcal{P}_{t-s} G_s\, ds \Big\| \leq \int_0^t \|G_s\| ds\,.$$

Given $g_1, g_2 \in \mathcal{A}^1$, by direct computation,

$$\mathcal{L}(g_1 g_2) - (\mathcal{L} g_1) g_2 - g_1 (\mathcal{L} g_2) = 2 \sum_{\alpha \in \mathcal{I}} [\ell_\alpha^*, g_1][g_2, \ell_\alpha]$$

whose operator norm, using (3.1), is bounded by $\sum_{x,y} \omega_{x,y} \delta_x(g_1) \delta_y(g_2)$. In view of Remark 3.6, the statement follows by choosing $g_i = \mathcal{P}_t f_i$, $i = 1, 2$. □

Proof of Proposition 3.7. Recalling (3.14), assumption (3.24) readily implies, for $s \geq 0$,

$$\sup_{y\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} \big(e^{s\Theta}\big)_{x,y} e^{\xi|x-y|} \leq e^{sM_\xi}.$$

This bound, together with assumption (3.25), yields

$$\sum_{x',y'\in\mathbb{Z}^d} \omega_{x',y'} \big(e^{s\Theta}\big)_{x',x} \big(e^{s\Theta}\big)_{y',y} \leq \Omega_\xi e^{2M_\xi s-\xi|x-y|}.$$

The statement now follows from Lemma 3.8 and elementary computations. $\square$

Proof of Theorem 2.3. To prove (i), we observe that for $f \in \mathcal{A}^1$ the sequence $(\mathcal{P}_t f)_{t\geq 0}$ is Cauchy in $\mathcal{A}$ as $t \to \infty$. Indeed, for $t \geq s \geq 0$

$$\mathcal{P}_t f - \mathcal{P}_s f = \int_s^t \mathcal{L}\mathcal{P}_r f \, dr,$$

so that

$$\begin{aligned} \|\mathcal{P}_t f - \mathcal{P}_s f\| &\leq \int_s^t \|\mathcal{L}\mathcal{P}_r f\| \, dr \leq C_0 \int_s^t |||\mathcal{P}_r f||| \, dr \\ &\leq C_0 \int_s^t e^{-(\lambda_1 - M)r} |||f||| \, dr = \frac{C_0}{\lambda_1 - M}\big(e^{-(\lambda_1-M)s} - e^{-(\lambda_1-M)t}\big)|||f|||\,, \end{aligned} \tag{3.26}$$

where we used Lemma 2.1 for the second inequality and Theorem 2.2(iv) for the third one. Let then $f_\infty = \lim_{t\to\infty} \mathcal{P}_t f \in \mathcal{A}$. We claim that $f_\infty \in \mathbb{C}\mathbf{1}$. Indeed, by Lemma 3.1 and Theorem 2.2(iv) $E_{x,h} f_\infty = 0$, $x \in \mathbb{Z}^d$, $h \in \{1, \dots, N\}$ which, as observed below (2.7), yields $f_\infty \in \mathbb{C}\mathbf{1}$.

Let $\pi\colon \mathcal{A}^1 \to \mathbb{C}$ be defined by $f_\infty = \pi(f)\mathbf{1}$. We claim that π extends to a state π on $\mathcal{A}$. The map π is obviously linear and, since $\mathcal{P}_t$ is a contraction, $|\pi(f)| = \lim_{t\to\infty} \|\mathcal{P}_t f\| \leq \|f\|$. Hence, $\|\pi\|_{\mathcal{A}'} \leq 1$. This bound implies that the map π can be extended to a continuous linear functional on $\mathcal{A}$. Furthermore $\pi(\mathbf{1}) = 1$. It remains to show that $\pi(f) \geq 0$ for $f \geq 0$. Since $(\mathcal{P}_t)_{t\geq 0}$ is a QMS, we have that $\mathcal{P}_t f \geq 0$, and passing to the limit $t \to \infty$ we obtain $f_\infty \geq 0$, which implies $\pi(f) \geq 0$. The semigroup property and the definition of π imply that $\pi(\mathcal{P}_t f) = \pi(f)$ for any $t \geq 0$ and $f \in \mathcal{A}^1$, hence π is a stationary state.

To show uniqueness, if ν is a stationary state, for $f \in \mathcal{A}^1$ we have $\nu(f) = \nu(\mathcal{P}_t f)$ which in the limit $t \to +\infty$ yields $\nu(f) = \nu(\pi(f)\mathbf{1}) = \pi(f)$. Hence $\nu = \pi$ since $\mathcal{A}^1$ is dense in $\mathcal{A}$.

To prove (ii) it suffices to take the limit $t \to \infty$ in (3.26).

Finally, we prove claim (iii). By the triangle inequality, for each $t \geq 0$,

$$\begin{aligned} |\pi(f_1 f_2) - \pi(f_1)\pi(f_2)| &\leq \|\mathcal{P}_t(f_1 f_2) - \pi(f_1 f_2)\mathbf{1}\| + \|\mathcal{P}_t f_1 - \pi(f_1)\mathbf{1}\| \|\mathcal{P}_t f_2\| \\ &\quad + |\pi(f_1)| \|\mathcal{P}_t f_2 - \pi(f_2)\mathbf{1}\| + \|\mathcal{P}_t(f_1 f_2) - (\mathcal{P}_t f_1)(\mathcal{P}_t f_2)\| \\ &\leq C e^{-(\lambda_1-M)t}\big(|||f_1 f_2||| + |||f_1||| \|f_2\| + \|f_1\| |||f_2|||\big) + \|\mathcal{P}_t(f_1 f_2) - (\mathcal{P}_t f_1)(\mathcal{P}_t f_2)\|\,, \end{aligned}$$

where we have used (ii) and have set $C = C_0/(\lambda_1 - M)$. The bound (3.12) implies

$$|||f_1 f_2||| \leq N\eta^2\big(|||f_1||| \|f_2\| + \|f_1\| |||f_2|||\big) \leq N\eta^2\big(\|f_1\| + |||f_1|||\big)\big(\|f_2\| + |||f_2|||\big).$$

Note that conditions (3.24) and (3.25) trivially hold for any $\xi > 0$ by the finite range assumptions. Moreover, denoting by R the range, we have $M_\xi \leq M e^{R\xi}$. We can thus apply Proposition 3.7 which yields

$$
\begin{aligned}
&|\pi(f_1 f_2) - \pi(f_1)\pi(f_2)| \\
&\le \frac{C_\xi}{2}\Big(e^{-(\lambda_1 - M)t} + e^{2Me^{R\xi}t - \xi\,\mathrm{dist}(\Lambda_1,\Lambda_2)}\Big)\big(\|f_1\| + |||f_1|||\big)\big(\|f_2\| + |||f_2|||\big),
\end{aligned}
$$

where $C_\xi = 2N\eta^2 \max\{C, \Omega_\xi/(2Me^{R\xi})\}$.

By choosing $t = \xi\big(\lambda_1 - M + 2Me^{R\xi}\big)^{-1}\mathrm{dist}(\Lambda_1,\Lambda_2)$, we deduce (iii) with $C = C_\xi$ and $\zeta = (\lambda_1 - M)\xi\big(\lambda_1 - M + 2Me^{R\xi}\big)^{-1}$. □

3.3. **Convergence of the specific quantum one-Wasserstein distance.** We first recall the dual formulation of the quantum one-Wasserstein distance W_Λ in terms of a Lipschitz seminorm proven in [14, Prop. 8]. Given $\Lambda \in \mathcal{P}$ we introduce the Lipschitz seminorm $|||\cdot|||_{\Lambda,\mathrm{Lip}}$ on $\mathcal{A}_\Lambda$ as $|||f|||_{\Lambda,\mathrm{Lip}} := \sup_{x\in\Lambda} \theta_x(f)$ where $\theta_x(f) := 2\inf_{g\in\mathcal{A}_{\Lambda\setminus\{x\}}} \|f - g\|$. Then the quantum one-Wasserstein distance on the set of states on $\mathcal{A}_\Lambda$ can be represented as

$$
W_\Lambda(\mu,\nu) = \sup_{|||f|||_{\Lambda,\mathrm{Lip}}\le 1} |\mu(f) - \nu(f)|. \tag{3.27}
$$

Lemma 3.9. *For each $\Lambda \in \mathcal{P}$ and $f \in \mathcal{A}_\Lambda$*

$$
|||f||| \le \frac{1}{2} N\eta|\Lambda|\, |||f|||_{\Lambda,\mathrm{Lip}}.
$$

Proof. Since $E_{x,h}f = E_{x,h}(f - g)$ for any $h \in \{1,\dots,N\}$ and $g \in \mathcal{A}_{\Lambda\setminus\{x\}}$, by Lemma 3.1, we deduce $\|E_{x,h}f\| \le \eta\theta_x(f)/2$. The statement follows. □

Proof of Theorem 2.4. Since $\mathcal{L}$ is translation covariant, the QMS $(\mathcal{P}_t)_{t\ge0}$ has a translation invariant stationary state π, which is unique by Theorem 2.3(i). Recalling (2.15), for each $\mu \in \mathcal{S}_\tau$ and $t \ge 0$,

$$
\begin{aligned}
w(\mu\mathcal{P}_t, \pi) &= \sup_{\Lambda\in\mathcal{P}} \frac{1}{|\Lambda|} \sup_{|||f|||_{\Lambda,\mathrm{Lip}}\le1} \big|\mu\big(\mathcal{P}_t f - \pi(f)\mathbf{1}\big)\big| \\
&\le \sup_{\Lambda\in\mathcal{P}} \frac{1}{|\Lambda|} \sup_{|||f|||_{\Lambda,\mathrm{Lip}}\le1} \big\|\mathcal{P}_t f - \pi(f)\mathbf{1}\big\|.
\end{aligned}
$$

The stated bound, with $C = C_0(\lambda_1 - M)^{-1}N\eta/2$, now follows directly from Theorem 2.3(ii) and Lemma 3.9. □

4. Interacting qudits: examples

To exemplify the abstract theory developed before, we next introduce simple dissipative quantum lattice systems and discuss when the perturbative criterion for ergodicity in Theorem 2.3 can be applied.

4.1. **Quantum spin systems: application of the perturbative criterion.** We consider a class of QMS with purely dissipative Lindblad generators $\mathcal{L}$ and show how the decomposition $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ can be achieved. We focus on the case of translation covariant interactions with finite range.

Let $H = \mathbb{C}^2$, $A = \mathcal{B}(\mathbb{C}^2)$ and denote by $\mathcal{H}$ and $\mathcal{A}$ the Hilbert space and the C^*-algebra constructed in Section 2.2. Let $\mathcal{I} = \mathbb{Z}^d \times \{1,2,3\}$ and $\chi\colon \mathcal{I} \to \mathcal{P}$ be the map $(x,j) \mapsto \{y\colon |x-y| \le R\} =: B_R(x)$. Consider the informal Lindblad generator

$$
\mathcal{L} = \sum_{\alpha\in\mathcal{I}} \big(\ell_\alpha^*[\,\cdot\,,\ell_\alpha] + [\ell_\alpha^*,\,\cdot\,]\ell_\alpha^*\big), \tag{4.1}
$$

where the jump operators ℓ_α satisfy $\ell_{(x,j)} \in \mathcal{A}_{B_R(x)}$ and are translation covariant. As we will next show, under suitable conditions on these operators, Theorems 2.2 and 2.3 can be applied to deduce that the graph norm closure of $\mathcal{L}$ generates an ergodic QMS.

We denote by σ_j, $j = 1, 2, 3$ the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and by $\sigma_{x,j}$, $j = 1, 2, 3$, the corresponding elements of $\mathcal{A}_{\{x\}}$.

Fix $x_0 \in \mathbb{Z}^d$, let $\mathcal{D}_{\{x_0\}} \subset \mathcal{A}_{\{x_0\}}$ be the real linear span of $\{\mathbf{1}, \sigma_{x_0,3}\}$ and set

$$\begin{aligned} \Delta_{1,2} &= \min\big\{ \|\sigma_{x_0,1} g - \ell_{(x_0,1)}\| + \|\sigma_{x_0,2} g - \ell_{(x_0,2)}\| \colon g \in \mathcal{D}_{\{x_0\}} \big\}, \\ \Delta_3 &= \min\big\{ \|g - \ell_{(x_0,3)}\| \colon g \in \mathcal{D}_{\{x_0\}} \big\}. \end{aligned} \tag{4.2}$$

Next, choose

$$\begin{aligned} a &\in \operatorname{argmin}\big\{ \|\sigma_{x_0,1} g - \ell_{(x_0,1)}\| + \|\sigma_{x_0,2} g - \ell_{(x_0,2)}\| \colon g \in \mathcal{D}_{\{x_0\}} \big\}, \\ b &\in \operatorname{argmin}\big\{ \|g - \ell_{(x_0,3)}\| \colon g \in \mathcal{D}_{\{x_0\}} \big\}. \end{aligned}$$

By translation covariance, $\Delta_{1,2}$, Δ_3 do not depend on x_0. Furthermore, identifying $\mathcal{A}_{\{x_0\}}$ with $A = \mathcal{B}(\mathbb{C}^2)$, there exist reals $\alpha_0, \alpha_1, \beta_0, \beta_1$ such that

$$a = \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_1 \end{pmatrix}, \qquad b = \begin{pmatrix} \beta_0 & 0 \\ 0 & \beta_1 \end{pmatrix}.$$

Hereafter, we assume $\alpha_0^2 + \alpha_1^2 > 0$.

Introduce on A the Lindblad generator

$$L_0 = \sum_{j=1}^{2} \big(a\sigma_j [\,\cdot\,, \sigma_j a] + [a\sigma_j, \,\cdot\,]\sigma_j a \big) + b\sigma_3 [\,\cdot\,, \sigma_3 b] + [b\sigma_3, \,\cdot\,]\sigma_3 b. \tag{4.3}$$

By direct computation, the Lindblad generator L_0 is self-adjoint with respect to the GNS inner product induced by the density matrix

$$\rho = \frac{1}{\alpha_0^2 + \alpha_1^2} \begin{pmatrix} \alpha_0^2 & 0 \\ 0 & \alpha_1^2 \end{pmatrix}.$$

The eigenvalues of $-L_0$ are $\lambda_0 = 0$, $\lambda = 4(\alpha_0^2 + \alpha_1^2)$, and $\mu = 2(\alpha_0^2 + \alpha_1^2) + (\beta_0 - \beta_1)^2$ with multiplicity 2. The corresponding normalized eigenvectors can be chosen as $e_0 = \mathbb{1}_{\mathbb{C}^2}$,

$$e_\lambda = \begin{pmatrix} \frac{\alpha_0}{\alpha_1} & 0 \\ 0 & -\frac{\alpha_1}{\alpha_0} \end{pmatrix}, \quad e_{\mu^+} = \sqrt{1 + \frac{\alpha_0^2}{\alpha_1^2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{\mu^-} = \sqrt{1 + \frac{\alpha_1^2}{\alpha_0^2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.4}$$

In particular, the spectral gap of $-L_0$ is $\lambda_1 = \lambda \wedge \mu$. Moreover, again by direct computation, $\eta = \max\{\|e_\lambda\|_A, \|e_{\mu^+}\|_A, \|e_{\mu^-}\|_A\} = (\alpha_0/\alpha_1) \vee (\alpha_1/\alpha_0)$.

The decomposition in Section 2.4 is then achieved as follows. Set $\mathcal{I}_0 = \mathcal{I} = \mathbb{Z}^d \times \{1, 2, 3\}$ and let $\imath \colon \mathcal{I}_0 \to \mathcal{I}$ be the identity map. Denoting by a_x, b_x the elements in $\mathcal{A}_{\{x\}}$ corresponding to a, b via the identification of A with $\mathcal{A}_{\{x\}}$, we then set $\ell^0_{(x,j)} = \sigma_{x,j} a_x$, $j = 1, 2$, $\ell^0_{(x,3)} = b_x$, and $\ell^1_\alpha = \ell_\alpha - \ell^0_\alpha$, $\alpha \in \mathcal{I}$. By construction

$$\mathcal{L} = \sum_{\alpha \in \mathcal{I}_0} \big({\ell^0_\alpha}^* [\,\cdot\,, \ell^0_\alpha] + [{\ell^0_\alpha}^*, \,\cdot\,] \ell^0_\alpha \big) + \sum_{\alpha \in \mathcal{I}} L^1_\alpha$$

in which L^1_α is given by the right-hand side of (2.12) with $k_\alpha = 0$.

Recalling (2.11) and (2.13), by few trite computations we get $C_0 = 4\eta \sum_{j=1}^3 \|\ell_{0,j}\|$ and

$$M \le 72\eta^2(\eta^2+1)(2R+1)^d \big(2(|\alpha_0| \vee |\alpha_1|)\Delta_{1,2} + 2(|\beta_0| \vee |\beta_1|)\Delta_3 + \Delta_{1,2}^2 + \Delta_3^2\big).$$

In particular, the QMS generated by $\mathcal{L}$ is ergodic if $\Delta_{1,2}$ and Δ_3 are small enough compared to the spectral gap λ_1.

4.2. **Quantum spin systems: conjugation with classical spin systems.** According to the terminology in [18, Ch.III], a (classical) spin system is a Markov semigroup $(\mathcal{P}_t^{\rm cl})_{t\ge 0}$ on the commutative C^*-algebra $C(\Omega)$ of the continuous $\mathbb{C}$-valued functions on $\Omega := \{-1,1\}^{\mathbb{Z}^d}$ whose generator acts on local functions by

$$(\mathcal{L}_{\rm cl} f_{\rm cl})(\sigma) = \sum_{x\in\mathbb{Z}^d} c_x(\sigma)[f_{\rm cl}(\sigma^x) - f_{\rm cl}(\sigma)], \qquad \sigma \in \Omega,$$

where $c_x\colon \Omega \to [0,\infty)$ is the flip rate and σ^x is the configuration in which the spin σ is flipped at site x. Many popular models like the stochastic Ising model, the contact process and the voter model are examples of spin systems. Provided the flip rates c_x satisfy suitable conditions, the semigroup generated by $\mathcal{L}_{\rm cl}$ is ergodic. On the other hand, for particular choices of the flip rates ergodicity fails, i.e. the stationary state is not unique. We refer to [18] for the details of both situations.

When the jump rates c_x have finite range, we next construct a QMS $(\mathcal{P}_t)_{t\ge0}$ whose action on a commutative subalgebra $\mathcal{D}$ of $\mathcal{A}$ is conjugate to the one of $(\mathcal{P}_t^{\rm cl})_{t\ge0}$, thus providing, for non-ergodic $(\mathcal{P}_t^{\rm cl})_{t\ge0}$, examples of non-ergodic QMS. The corresponding Lindblad generator has the form considered in the previous section, see in particular (4.1). Let $\mathcal{D} \subset \mathcal{A}$ be the commutative C^*-subalgebra of $\mathcal{A}$ generated by $\sigma_{x,3}$, $x \in \mathbb{Z}^d$. Consider the C^*-algebra isomorphism $\imath\colon C(\Omega) \to \mathcal{D}$ defined by $\imath(f_{\rm cl}) = f_{\rm cl}(\{\sigma_{x,3}\}_{x\in\mathbb{Z}^d})$. By choosing $\ell_{x,j} = \imath(\sqrt{c_x})\sigma_{x,j}$, $j = 1,2$ and $\ell_{x,3} \in \mathcal{D}$, a direct computation shows that $\mathcal{L}\circ\imath = \imath\circ\mathcal{L}_{\rm cl}$. Hence the QMS $(\mathcal{P}_t)_{t\ge0}$ generated by $\mathcal{L}$ leaves invariant $\mathcal{D}$ and its action on $\mathcal{D}$ is conjugated to the one of $(\mathcal{P}_t^{\rm cl})_{t\ge0}$.

4.3. ***XYZ*-model with site dissipation.** In this section we consider a Heisenberg perturbation induced by the XYZ-Hamiltonian of non-interacting dissipative spins. We show that, if the interaction parameters are small enough the resulting evolution is ergodic.

As in the previous sections, let $H = \mathbb{C}^2$, $A = \mathcal{B}(H)$, and σ_j, $j = 1,2,3$ be the Pauli matrices. The one-qubit unperturbed Lindblad generator is

$$L_0 = \frac{1}{4}\sum_{j=1}^{2}\big(\sigma_j[\,\cdot\,,\sigma_j] + [\sigma_j,\,\cdot\,]\sigma_j\big)$$

which is self-adjoint with respect to the GNS inner product induced by the state $\rho = (1/2)\mathbb{1}_H$. Note that this generator is a particular case of the one introduced in (4.3) when $a = \mathbb{1}_H$ and $b = 0$. In particular, the eigenvalues of $-L_0$ are $\lambda_0 = 0$, $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = 4$, with corresponding normalized eigenvectors given by (4.4) with $\alpha_0 = \alpha_1 = 1/2$ and $\beta_0 = \beta_1 = 0$

$$e_0 = \mathbb{1}_H, \quad e_1 = \sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \sigma_3.$$

Hence $\eta = \max_{j\in\{1,2,3\}} \|e_j\|_{H\to H} = \sqrt{2}$. As in Section 2.2, we denote by $\mathcal{L}_0$ the Lindblad generator in which each qubit evolves independently according to L_0, see equation (2.4).

Given $J_j \in \mathbb{R}$, $j = 1,2,3$, let $k_{X,j} = J_j\sigma_{x,j}\sigma_{y,j}$ if $X = \{x,y\}$ with $|x-y| = 1$, and $k_{X,j} = 0$ otherwise. Here we understand, as in the previous section, $\sigma_{x,j} = \sigma_j \otimes \mathbb{1}_{\mathcal{H}_{\{x\}^c}}$, $j = 1,2,3$. The XYZ model is then defined in terms of the informal Hamiltonian $\mathcal{K} = \sum_{X\in\mathcal{P}}\sum_{j=1}^{3} k_{X,j}$. Accordingly, we set $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ where $\mathcal{L}_1$ is the informal Heisenberg operator $\mathcal{L}_1 = i[\mathcal{K},\cdot]$. To fit this case in the general framework of Section 2.3, set $\mathcal{I} = \mathcal{P}\times\{1,2,3\}$, let $\chi\colon \mathcal{I}\to\mathcal{P}$ be the projection on the first coordinate, k_α as defined above, $\ell_{(\{x\},j)} = (1/2)\sigma_{x,j}$, $x \in \mathbb{Z}^d$, $j = 1,2$, and $\ell_\alpha = 0$ otherwise. Note that the family $\{k_\alpha, \ell_\alpha\}_{\alpha\in\mathcal{I}}$ is translation covariant and has range $R = 1$.

As by definition $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$, to apply the results of Section 2.4 it is enough to set $\mathcal{I}_0 = \mathbb{Z}^d \times \{1,2\}$ with $\imath\colon \mathcal{I}_0 \to \mathcal{I}$ given by $\imath(x,j) = (\{x\},j)$. By direct computation $\|k_{(\{x,y\},j)}\| \le |J_j|$ for $|x-y| = 1$ and therefore the constant in (2.11) satisfies $C_0 \le 2\sqrt{2}(1+2d|J|)$, where $|J| = \sum_{j=1}^{3}|J_j|$. Recalling (2.13), few trite computations yield $M \le 96\sqrt{2}d|J|$. In particular, by Theorem 2.3, the QMS generated by $\mathcal{L}$ is ergodic whenever $|J| < (48\sqrt{2}d)^{-1}$.

5. Interacting fermions

In this section we consider quantum lattice systems described in terms of fermionic operators satisfying the *canonical anticommutation relations* (CAR). The unperturbed dynamics is given by the Fermi Ornstein-Uhlenbeck semigroup, while the interaction will be expressed as a superposition of local generators.

5.1. Canonical anticommutation relations and fermionic C^*-algebra.

Referring e.g. to [17] for the abstract setting of Clifford algebras, we next introduce a family of operators satisfying the CAR in a concrete representation that describes fermions on the whole lattice $\mathbb{Z}^d$.

Let $\mathcal{H}$ be the Hilbert space with complete orthonormal system $\{e_X\}_{X\in\mathcal{P}}$, where we recall that $\mathcal{P}$ denotes the family of the finite subsets of $\mathbb{Z}^d$. For $\Lambda \subset \mathbb{Z}^d$ we also consider the subspace $\mathcal{H}_\Lambda$ spanned by $\{e_X\}_{X\subset\Lambda}$. If Λ is finite, then $\mathcal{H}_\Lambda$ has dimension $2^{|\Lambda|}$. Clearly, $\mathcal{H}^0 = \bigcup_{\Lambda\in\mathcal{P}}\mathcal{H}_\Lambda$ is dense in $\mathcal{H}$. Moreover $\mathcal{H}_{\Lambda_1\sqcup\Lambda_2} \simeq \mathcal{H}_{\Lambda_1}\otimes\mathcal{H}_{\Lambda_2}$, where $\sqcup$ denotes the union of disjoint sets.

Denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of bounded operators on $\mathcal{H}$ and by $\|\cdot\|$ the corresponding norm. As in the previous sections $\mathbf{1} \in \mathcal{B}(\mathcal{H})$ is the identity. Fix a total order $\le$ on $\mathbb{Z}^d$ and for $x \in \mathbb{Z}^d$ define $a_x \in \mathcal{B}(\mathcal{H})$ by

$$a_x e_X = \mathbf{1}_X(x)(-1)^{|\{y\in X\,:\, y<x\}|} e_{X\setminus\{x\}}, \qquad X \in \mathcal{P}$$

where $\mathbf{1}_X$ denotes the indicator function of the set X. Accordingly,

$$a_x^* e_X = \mathbf{1}_{\mathbb{Z}^d\setminus X}(x)(-1)^{|\{y\in X\,:\, y<x\}|} e_{X\cup\{x\}}, \qquad X \in \mathcal{P}.$$

By direct computations, the family $\{a_x, a_x^*\}_{x\in\mathbb{Z}^d}$ satisfies the CAR, i.e.

$$\{a_x, a_y\} = \{a_x^*, a_y^*\} = 0, \qquad \{a_x, a_y^*\} = \delta_{x,y}\mathbf{1} \tag{5.1}$$

where $\{a,b\} = ab + ba$ is the anticommutator of a and b. Let $n_x = a_x^* a_x$, $x \in \mathbb{Z}^d$, be the fermionic number operators. These operators are pairwise commuting, self-adjoint, satisfy $n_x^2 = n_x$, and act on $\mathcal{H}$ as $n_x e_X = \mathbf{1}_X(x) e_X$.

For $\Lambda \in \mathcal{P}$, let $\mathcal{A}_\Lambda \subset \mathcal{B}(\mathcal{H})$ be the subalgebra generated by $\{a_x, a_x^*\}_{x\in\Lambda}$ and set $\mathcal{A}^0 := \bigcup_{\Lambda\in\mathcal{P}} \mathcal{A}_\Lambda$. Noticing that $\mathcal{A}^0$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, we finally let $\mathcal{A}$ be the norm closure of $\mathcal{A}^0$. In particular, $\mathcal{A}$ is a unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. In fact, it is a proper subalgebra as, for example, the translation operators τ_x, $x \in \mathbb{Z}^d$, defined by $\tau_x(e_X) = e_{X+x}$, lie in $\mathcal{B}(\mathcal{H})$ but not in $\mathcal{A}$. As in the previous section, we denote by $\mathcal{S}$ the collection of states on $\mathcal{A}$.

Note that the Hilbert space $\mathcal{H}$ constructed above can be identified with the one presented in Section 2.2 when $H = \mathbb{C}^2$. However, under this identification, the C^*-algebra $\mathcal{A}$ defined here does not coincide with the C^*-algebra $\mathcal{A}$ defined in Section 2.2, since the fermionic operators a_x, a_x^* are not local.

In order to define the dynamics, we next introduce another family of operators $\{v_x, v_x^*\}_{x\in\mathbb{Z}^d}$ satisfying the CAR. They will have the property that v_x and a_y commute for $x \neq y$. To this end, let w be the self-adjoint and unitary element of $\mathcal{B}(\mathcal{H})$ given by

$$we_X = (-1)^{|X|} e_X, \qquad X \in \mathcal{P}.$$

The operator w is usually referred to as the main automorphism or sign operator [17]. Observe that it does not belong to $\mathcal{A}$ and

$$wa_x = -a_x w, \qquad wa_x^* = -a_x^* w, \qquad x \in \mathbb{Z}^d. \tag{5.2}$$

Hence, letting $\mathrm{Ad}_w(f) = wfw$, $f \in \mathcal{B}(\mathcal{H})$, for each $\Lambda \in \mathcal{P}$, the subalgebra $\mathcal{A}_\Lambda$ is left invariant by Ad_w. In particular, Ad_w defines an outer automorphism of $\mathcal{A}$.

Define also the operators in $w\mathcal{A} \subset \mathcal{B}(\mathcal{H})$

$$v_x = wa_x\ , \qquad v_x^* = a_x^* w\,, \qquad x \in \mathbb{Z}^d\,. \tag{5.3}$$

Readily, also the family $\{v_x, v_x^*\}_{x\in\mathbb{Z}^d}$ satisfies the CAR. Moreover, by direct computations,

$$[v_x, a_y] = [v_x^*, a_y^*] = 0, \qquad [v_x, a_y^*] = -[v_x^*, a_y] = \delta_{x,y} w\,, \qquad x, y \in \mathbb{Z}^d\,. \tag{5.4}$$

For $\Lambda \in \mathcal{P}$ we let $\mathcal{V}_\Lambda$ be the finite-dimensional C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\{v_x,\, v_x^*\}_{x\in\Lambda}$. As for the algebra $\mathcal{A}$, set $\mathcal{V}^0 := \bigcup_{\Lambda\in\mathcal{P}} \mathcal{V}_\Lambda \subset \mathcal{B}(\mathcal{H})$, and let $\mathcal{V}$ be its norm closure. As follows from (5.4), for disjoint $X, Y \in \mathcal{P}$ the algebras $\mathcal{V}_X$ and $\mathcal{A}_Y$ commute, that is

$$[u, f] = 0, \quad \text{for any } u \in \mathcal{V}_X,\ f \in \mathcal{A}_Y \text{ with } X \cap Y = \emptyset. \tag{5.5}$$

For $\Lambda \in \mathcal{P}$, the algebras $\mathcal{A}_\Lambda$ and $\mathcal{V}_\Lambda$ are $\mathbb{Z}/2\mathbb{Z}$-graded, by letting $\deg(a_x) = \deg(a_x^*) = 1$ and $\deg(v_x) = \deg(v_x^*) = 1$, respectively. Hence, we have the decompositions $\mathcal{A}_\Lambda = \mathcal{A}_{\Lambda,0} \oplus \mathcal{A}_{\Lambda,1}$ and $\mathcal{V}_\Lambda = \mathcal{V}_{\Lambda,0} \oplus \mathcal{V}_{\Lambda,1}$, where $\mathcal{A}_{\Lambda,p}$ and $\mathcal{V}_{\Lambda,p}$ are the subspaces of parity $p \in \mathbb{Z}/2\mathbb{Z}$, respectively. In view of (5.2), $\mathcal{V}_{\Lambda,0} = \mathcal{A}_{\Lambda,0}$ and $\mathcal{V}_{\Lambda,1} = w\mathcal{A}_{\Lambda,1}$.

Following [8, 12] we introduce a gradient structure induced by the fermionic operators. Let ∂_x, $\bar{\partial}_x$, $x \in \mathbb{Z}^d$, be the bounded operators on $\mathcal{A}$ defined by

$$\partial_x := w[v_x, \cdot\,]\ , \qquad \bar{\partial}_x := -w[v_x^*, \cdot\,]\,. \tag{5.6}$$

In view of (5.2) and (5.3), if $f \in \mathcal{A}_{\{x\}^c}$ then $\partial_x f = \bar{\partial}_x f = 0$. Moreover, ∂_x and $\bar{\partial}_x$ are skew derivations in the sense that for each $f, g \in \mathcal{A}$

$$\partial_x(fg) = (\partial_x f)g + \mathrm{Ad}_w(f)\partial_x g, \qquad \bar{\partial}_x(fg) = (\bar{\partial}_x f)g + \mathrm{Ad}_w(f)\bar{\partial}_x g.$$

Let also $\check{\partial}_x$, $\check{\bar{\partial}}_x$, $x \in \mathbb{Z}^d$, be the bounded operators on $\mathcal{V}$ defined by

$$\check{\partial}_x := w[a_x, \cdot\,]\ , \qquad \check{\bar{\partial}}_x := -w[a_x^*, \cdot\,]\,. \tag{5.7}$$

5.2. **Dynamics.** As unperturbed dynamics we consider the Fermi Ornstein-Uhlenbeck semigroup introduced in [8, 12], that describe the dissipative evolution of free fermions.

For $h \in \mathbb{R}$ let $\pi_0 \in \mathcal{S}$ be the product state corresponding to free fermions with external field h. More precisely, for $\Lambda \in \mathcal{P}$ and $f \in \mathcal{A}_\Lambda$, we set

$$\pi_0(f) = \frac{\mathrm{Tr}_{\mathcal{H}_\Lambda}\left(e^{h\sum_{x\in\Lambda} n_x} f\right)}{(1+e^h)^{|\Lambda|}}$$

which, by the density of $\mathcal{A}^0$ in $\mathcal{A}$, uniquely defines π_0.

We then define the unperturbed generator on $\mathcal{A}$ by

$$\mathcal{L}_0 = \sum_{x\in\mathbb{Z}^d} L_x^0 \tag{5.8}$$

where

$$L_x^0 = e^{h/2}\big([v_x,\,\cdot\,]v_x^* + v_x[\,\cdot\,,v_x^*]\big) + e^{-h/2}\big([v_x^*,\,\cdot\,]v_x + v_x^*[\,\cdot\,,v_x]\big). \tag{5.9}$$

By direct computation, $L_x^0\mathcal{A} \subset \mathcal{A}$, and $\mathcal{L}_0$ is symmetric with respect to GNS inner product induced by π_0.

In this section, we define the seminorm $|||\cdot|||$ on $\mathcal{A}^0$ by

$$|||f||| := \sum_{x\in\mathbb{Z}^d} (\|\partial_x f\| + \|\bar\partial_x f\|). \tag{5.10}$$

Observe that $|||f||| = 0$ if and only if f is a scalar multiple of $\mathbf{1}$. Let also $\mathcal{A}^1$ be the closure of $\mathcal{A}^0$ with respect to the norm $\|\cdot\| + |||\cdot|||$. Clearly, $\mathcal{A}^0 \subset \mathcal{A}^1 \subset \mathcal{A}$.

We consider perturbations of the generator $\mathcal{L}_0$, both of conservative and dissipative type. The local Hamiltonians will be assumed, as natural from a physical viewpoint, to be even functions of the fermionic operators, while the jump operators associated to the dissipative perturbation, belonging to $\mathcal{V}$, can be both even and odd but we will require them to have a definite parity. More precisely, fix a countable set $\mathcal{I}$ and functions $p\colon \mathcal{I} \to \mathbb{Z}/2\mathbb{Z}$ and $\chi\colon \mathcal{I} \to \mathcal{P}$ with finite fibers $\chi^{-1}(X)$, $X \in \mathcal{P}$. Denote also $\mathcal{I}_0 = p^{-1}(0)$ and $\mathcal{I}_1 = p^{-1}(1)$. Fix then a collection $\{k_\alpha, \ell_\alpha\}_{\alpha\in\mathcal{I}}$ such that $k_\alpha = k_\alpha^* \in \mathcal{V}_{\chi(\alpha),0} = \mathcal{A}_{\chi(\alpha),0}$ for $\alpha \in \mathcal{I}_0$, $k_\alpha = 0$ for $\alpha \in \mathcal{I}_1$, and $\ell_\alpha \in \mathcal{V}_{\chi(\alpha),p(\alpha)}$ for every $\alpha \in \mathcal{I}$. We then set

$$\mathcal{L}_1 = \sum_{\alpha\in\mathcal{I}} L_\alpha^1, \quad \text{where} \quad L_\alpha^1 = i[k_\alpha,\,\cdot\,] + [\ell_\alpha^*,\,\cdot\,]\ell_\alpha + \ell_\alpha^*[\,\cdot\,,\ell_\alpha]\,. \tag{5.11}$$

The above parity assumptions guarantee that $L_\alpha^1\mathcal{A} \subset \mathcal{A}$, $\alpha \in \mathcal{I}$. We will show in Lemma 5.1 and Theorem 5.2 that, under suitable conditions on the family $\{k_\alpha, \ell_\alpha\}_{\alpha\in\mathcal{I}}$, the right-hand side of (5.11) is well defined on $\mathcal{A}^1$, and the graph norm closure of $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ generates a QMS on $\mathcal{A}$.

As in Section 2, the family $\big\{k_\alpha, \ell_\alpha\big\}_{\alpha\in\mathcal{I}}$, has *finite range* if there exists $R \in [0,\infty)$ such that $k_\alpha = \ell_\alpha = 0$ whenever $\mathrm{diam}(\chi(\alpha)) > R$. The family $\big\{k_\alpha, \ell_\alpha\big\}_{\alpha\in\mathcal{I}}$ is *translation covariant* if there exists an action of the abelian group $\mathbb{Z}^d$ on $\mathcal{I}$, denoted by $(x, \alpha) \mapsto x + \alpha$, satisfying $\chi(x + \alpha) = x + \chi(\alpha)$, such that $\mathrm{Ad}_{\tau_x}(k_\alpha) = k_{x+\alpha}$ and $\mathrm{Ad}_{\tau_x}(\ell_\alpha) = \ell_{x+\alpha}$.

5.3. **Main results.** As we next state, under suitable assumptions, the operator $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ is well defined on $\mathcal{A}^1$.

Lemma 5.1. *If*

$$C_0 := 2\,\mathrm{ch}(h/2) + 4 \sup_{x\in\mathbb{Z}^d} \sum_{\alpha\colon \chi(\alpha)\ni x} (\|k_\alpha\| + 2\|\ell_\alpha\|^2) < +\infty, \tag{5.12}$$

then for each $f \in \mathcal{A}^1$ the series defining $\mathcal{L}f$ converges in $\mathcal{A}$ and $\|\mathcal{L}f\| \le C_0 |||f|||$.

As in the case of qudits, we first show the existence of the dynamics of interacting fermions. For $\Lambda \in \mathcal{P}$ we denote by $\mathcal{L}_\Lambda$ the bounded Lindblad generator on $\mathcal{A}$ defined by

$$\mathcal{L}_\Lambda := \sum_{x\in\Lambda} L^0_x + \sum_{\alpha\in\mathcal{I}\colon \chi(\alpha)\subset\Lambda} L^1_\alpha$$

and by $(\mathcal{P}^\Lambda_t)_{t\ge 0}$ the corresponding QMS.

Theorem 5.2. *Assume* (5.12) *and consider $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ as an operator on $\mathcal{A}$ with domain $\mathcal{A}^1$. If*

$$\begin{aligned} M :=& 4 \sup_{y\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} \sum_{\alpha\colon \chi(\alpha)\ni y} \Big(\|\partial_x k_\alpha\| + \|\bar{\partial}_x k_\alpha\| \\ &+ 2\|\ell_\alpha\| \big(\|\check{\partial}_x \ell_\alpha\| + \|\check{\partial}_x \ell^*_\alpha\| + \|\check{\bar{\partial}}_x \ell_\alpha\| + \|\check{\bar{\partial}}_x \ell^*_\alpha\| \big) \Big) < \infty, \end{aligned} \tag{5.13}$$

then

(i) the graph norm closure $\bar{\mathcal{L}}$ of $\mathcal{L}$ generates a QMS $(\mathcal{P}_t)_{t\ge0}$ on $\mathcal{A}$;
(ii) $\mathcal{A}^1$ is a core for $\bar{\mathcal{L}}$;
(iii) for each $t \ge 0$ the operator $\mathcal{P}_t$ is the strong limit of $\mathcal{P}^\Lambda_t$ as $\Lambda \uparrow \mathbb{Z}^d$;
(iv) for any $f \in \mathcal{A}^1$ and $t \ge 0$ we have

$$|||\mathcal{P}_t f||| \le e^{(M - 2\,\mathrm{ch}(h/2))t} |||f|||\,;$$

(v) the QMS $(\mathcal{P}_t)_{t\ge0}$ has at least one stationary state.

The perturbative criterion for the ergodicity of interacting fermions is then stated as follows.

Theorem 5.3. *Assume* (5.12) *and $M < 2\,\mathrm{ch}(h/2)$. Then*

(i) the QMS $(\mathcal{P}_t)_{t\ge0}$ has a unique stationary state π;
(ii) for any $f \in \mathcal{A}^1$ and $t \ge 0$

$$\|\mathcal{P}_t f - \pi(f)\mathbf{1}\| \le \frac{C_0}{2\,\mathrm{ch}(h/2) - M} e^{-(2\,\mathrm{ch}(h/2) - M)t} |||f|||\,;$$

(iii) if furthermore $\{k_\alpha, \ell_\alpha\}_{\alpha\in\mathcal{I}}$ has finite range, there exist $C, \zeta > 0$ such that for any $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$ and any $f_1 \in \mathcal{A}_{\Lambda_1}$, $f_2 \in \mathcal{A}_{\Lambda_2}$,

$$|\pi(f_1 f_2) - \pi(f_1)\pi(f_2)| \le C e^{-\zeta\,\mathrm{dist}(\Lambda_1,\Lambda_2)} \big(\|f_1\| + |||f_1|||\big)\big(\|f_2\| + |||f_2|||\big)\,.$$

As $\mathrm{gap}(-L^0_x) = 2\,\mathrm{ch}(h/2)$, the above criterion corresponds to the one formulated in Theorem 2.3 for qudits. As in Section 2, in the translation covariant case, Theorem 5.3 implies the exponential decay of specific quantum one-Wasserstein distance between $\mu\mathcal{P}_t$ and π. We refer to Theorem 2.4 for the precise statement.

5.4. **Bounds on the commutators.** The proofs of Theorems 5.2 and 5.3 are accomplished by the arguments presented in Section 3. The relevant ingredients are: (i) the intertwining relationships, as discussed in [8] and stated in Lemma 5.5 below, between the derivatives ∂_x, $\bar\partial_x$ and the unperturbed generator $\mathcal{L}_0$; (ii) some quantitative bounds on the commutators between the derivatives ∂_x, $\bar\partial_x$ and the perturbed generator $\mathcal{L}_1$ that are here derived anew.

We start with the following lemma which provides the fermionic counterpart to (2.6).

Lemma 5.4. *Let E_x be the normalized partial trace on $\mathcal{H}_{\{x\}}$ and set*

$$D_x = a_x^*\partial_x = v_x^*[v_x,\,\cdot\,], \qquad \bar D_x = a_x\bar\partial_x = v_x[v_x^*,\,\cdot\,],$$

that are regarded as bounded operators on $\mathcal{A}$. Then for each $x \in \mathbb{Z}^d$ and $f \in \mathcal{A}$

$$f = E_x f + D_x f + \bar D_x f - \frac{1}{2}\big(D_x\bar D_x + \bar D_x D_x\big)f. \tag{5.14}$$

Proof. By linearity and density it suffices to show (5.14) when f is a monomial; namely, when $f = \otimes_y f_y$ with $f_y \in \mathcal{A}_{\{y\}}$ and the product runs over finitely many sites. In view of (5.4), for such f

$$D_x f = \big(\otimes_{y<x} f_y\big)(D_x f_x)\big(\otimes_{y>x} f_y\big), \quad \bar D_x f = \big(\otimes_{y<x} f_y\big)(\bar D_x f_x)\big(\otimes_{y>x} f_y\big).$$

On the other hand, by direct computations,

$$\begin{array}{llll} D_x\mathbf{1} = 0, & D_x a_x = 0, & D_x a_x^* = a_x^*, & D_x n_x = n_x, \\ \bar D_x\mathbf{1} = 0, & \bar D_x a_x = a_x, & \bar D_x a_x^* = 0, & \bar D_x n_x = n_x - \mathbf{1}. \end{array}$$

As the linear span of $\{\mathbf{1}, a_x, a_x^*, n_x\}$ is $\mathcal{A}_{\{x\}}$, the statement follows by linearity. □

Proof of Lemma 5.1. Since $\|w\| = \|a_x\| = \|a_x^*\| = 1$ and $w^2 = \mathbf{1}$, we have

$$\|[v_x, f]\| = \|\partial_x f\|\,, \quad \|[v_x^*, f]\| = \|\bar\partial_x f\|\,, \quad \|[n_x, f]\| \le \|\partial_x f\| + \|\bar\partial_x f\|\,. \tag{5.15}$$

Hence, from (5.9) we get

$$\|L_x^0 f\| \le 2\,\mathrm{ch}(h/2)\big(\|\partial_x f\| + \|\bar\partial_x f\|\big). \tag{5.16}$$

As follows from a direct computation, the statement is achieved by the following estimate. For each $X \in \mathcal{P}$, $u \in \mathcal{V}_X$, and $f \in \mathcal{A}^1$

$$\|[u, f]\| \le 4\|u\| \sum_{x\in X} (\|\partial_x f\| + \|\bar\partial_x f\|). \tag{5.17}$$

To prove the bound (5.17), let $X = \{x_1, \ldots, x_m\}$ and introduce the operator $F_x\colon \mathcal{A} \to \mathcal{A}$ defined by $F_x = D_x + \bar D_x - (1/2)(D_x\bar D_x + \bar D_x D_x)$. By recursively using (5.14) we deduce, cf. (3.3),

$$f = \Big(\prod_{j=1}^m E_{x_j}\Big) f + \sum_{j=1}^m \Big(\prod_{h=1}^{j-1} E_{x_h}\Big) F_{x_j} f. \tag{5.18}$$

Since $\|E_x\|_{\mathcal{A}\to\mathcal{A}} \le 1$, $\|D_x\|_{\mathcal{A}\to\mathcal{A}} \le 2$, $\|\bar D_x\|_{\mathcal{A}\to\mathcal{A}} \le 2$, $\|D_x f\| \le \|\partial_x f\|$, and $\|\bar D_x f\| \le \|\bar\partial_x f\|$, we deduce

$$\Big\|\Big(\prod_{h=1}^{j-1} E_{x_h}\Big) F_{x_j} f\Big\| \le 2\big(\|\partial_{x_j} f\| + \|\bar\partial_{x_j} f\|\big), \qquad j = 1, \ldots, m. \tag{5.19}$$

Recalling (5.5), the claim (5.17) follows by noticing that $\big(\prod_{j=1}^m E_{x_j}\big) f$ belongs to $\mathcal{A}_{X^c}$. □

We next recall the intertwining relationship between the unperturbed generator $\mathcal{L}_0$ and the gradient structure introduced in (5.6).

Lemma 5.5. *For each $f \in \mathcal{A}^1$ and $x \in \mathbb{Z}^d$*

$$\partial_x \mathcal{L}_0 f - \mathcal{L}_0 \partial_x f = -2\,\mathrm{ch}(h/2)\partial_x f$$
$$\bar{\partial}_x \mathcal{L}_0 u - \mathcal{L}_0 \bar{\partial}_x f = -2\,\mathrm{ch}(h/2)\bar{\partial}_x f.$$

Proof. Both identities are obtained by a straightforward computation, see also [8, §6.2]. □

Lemma 5.6. *Fix $x \in \mathbb{Z}^d$ and $X \in \mathcal{P}$.*

(i) For each $u \in \mathcal{V}_{X,0}$ and $f \in \mathcal{A}^1$

$$\|\partial_x[u,f] - [u,\partial_x f]\| \le 4\|\partial_x u\| \sum_{y\in X} \big(\|\partial_y f\| + \|\bar{\partial}_y f\|\big),$$
$$\|\bar{\partial}_x[u,f] - [u,\bar{\partial}_x f]\| \le 4\|\bar{\partial}_x u\| \sum_{y\in X} \big(\|\partial_y f\| + \|\bar{\partial}_y f\|\big).$$

(ii) For each $j = 0, 1$, $u \in \mathcal{V}_{X,j}$, and $f \in \mathcal{A}^1$

$$\big\|\partial_x(u^*[f,u] + [u^*,f]u) - u^*[\partial_x f,u] - [u^*\partial_x f]u\big\| \le 8\|u\|\big(\|\check{\partial}_x u^*\| + \|\check{\partial}_x u\|\big) \sum_{y\in X} \big(\|\partial_y f\| + \|\bar{\partial}_y f\|\big),$$
$$\big\|\bar{\partial}_x(u^*[f,u] + [u^*,f]u) - u^*[\bar{\partial}_x f,u] - [u^*\bar{\partial}_x f]u\big\| \le 8\|u\|\big(\|\check{\bar{\partial}}_x u^*\| + \|\check{\bar{\partial}}_x u\|\big) \sum_{y\in X} \big(\|\partial_y f\| + \|\bar{\partial}_y f\|\big).$$

Proof. We use the notation introduced in the proof of Lemma 5.1. For (i), we prove only the first bound. Since $u \in \mathcal{V}_{X,0}$, the Jacobi identity yields

$$\partial_x[u,f] - [u,\partial_x f] = w[[v_x,u],f] = w\sum_{j=1}^m \Big[[v_x,u], \Big(\prod_{h=1}^{j-1} E_{x_h}\Big) F_{x_j} f\Big],$$

where we used (5.18) and (5.5) in the second step. The statement now follows from (5.19).

Regarding (ii), we prove again only the first bound. By direct computation, for $j = 0, 1$, $u \in \mathcal{V}_j$, $f \in \mathcal{A}$,

$$\partial_x(u^*[f,u]) - u^*[\partial_x f,u] = w\big((\check{\partial}_x u^*)[f,u] + (-1)^j u^*[f,\check{\partial}_x u]\big)$$
$$\partial_x([u^*,f]u) - [u^*,\partial_x f]u = w\big([\check{\partial}_x u^*,f]u + (-1)^j [u^*,f](\check{\partial}_x u)\big).$$

The statement now follows by (5.17). □

As we next state, the estimates in Lemma 5.6 provide the required bounds on the commutator between the gradient structure introduced in (5.6) and perturbed generator $\mathcal{L}_1$.

Lemma 5.7. *For $x, y \in \mathbb{Z}^d$ set*

$$\theta_{x,y} := 4 \sum_{\alpha:\, y\in\chi(\alpha)} \big(\|\partial_x k_\alpha\| + 2\|\ell_\alpha\|(\|\check{\partial}_x \ell_\alpha\| + \|\check{\partial}_x \ell^*_\alpha\|)\big)$$

$$\widetilde{\theta}_{x,y} := 4 \sum_{\alpha:\, y\in\chi(\alpha)} \big(\|\bar{\partial}_x k_\alpha\| + 2\|\ell_\alpha\|(\|\check{\bar{\partial}}_x \ell_\alpha\| + \|\check{\bar{\partial}}_x \ell^*_\alpha\|)\big).$$

Then for each $x \in \mathbb{Z}^d$ and $f \in \mathcal{A}^1$

$$\|\partial_x \mathcal{L}_1 f - \mathcal{L}_1 \partial_x f\| \le \sum_{y\in\mathbb{Z}^d} \theta_{x,y}(\|\partial_y f\| + \|\bar{\partial}_y f\|)\,,$$

$$\|\bar{\partial}_x \mathcal{L}_1 f - \mathcal{L}_1 \bar{\partial}_x f\| \le \sum_{y\in\mathbb{Z}^d} \widetilde{\theta}_{x,y}(\|\partial_y f\| + \|\bar{\partial}_y f\|).$$

Proof. The result is a direct consequence of Lemma 5.6. □

Recalling (5.13), note that

$$M = \sup_{y\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} (\theta_{x,y} + \widetilde{\theta}_{x,y}).$$

Lemmata 5.1, 5.5, and 5.7 provide the ingredients to achieve the proofs of Theorems 5.2 and 5.3 by the arguments presented in Section 3.

5.5. **Nearest neighbor interacting fermions with site dissipation.** Consider the informal Hamiltonian given by

$$\mathcal{K} = J \sum_{\{x,y\}:\, |x-y|=1} \big(a^*_x a_y + a^*_y a_x\big),$$

for some $J \in \mathbb{R}$. Letting $\mathcal{L}_0$ be the Lindblad generator of the Fermi Ornstein-Uhlenbeck QMS introduced in (5.8), consider the QMS with informal generator

$$\mathcal{L} = \mathcal{L}_0 + i[\mathcal{K}, \cdot\,].$$

It fits the setting introduced in Section 5.2 with the translation covariant conservative interaction parametrized by $\mathcal{I} = \big\{\{x,y\}:\, x, y \in \mathbb{Z}^d, |x-y| = 1\big\} \subset \mathcal{P}$ and given by

$$k_{\{x,y\}} = J\big(a^*_x a_y + a^*_y a_x\big),$$

understanding that $\chi\colon \mathcal{I} \to \mathcal{P}$ is the inclusion map. In particular, the family $\{k_\alpha\}_{\alpha\in\mathcal{I}}$ has range 1. By direct computation, $\|k_{\{x,y\}}\| = |J|$ and

$$\theta_{x,y} = \widetilde{\theta}_{x,y} = \begin{cases} 8|J|d & \text{if } x = y, \\ 4|J| & \text{if } |x-y| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Correspondingly, $M = 32d|J|$, so that the assumptions of Theorem 5.3 are met when $|J| < \mathrm{ch}(h/2)/(16d)$.

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References

[1] R. Alicki, *On the detailed balance conditions for non-Hamiltonian systems.* Rep. Math. Phys. **10** (1976), 249–258.

[2] R. Alicki, K. Lendi, *Quantum dynamical semigroups and applications.* Second edition. Lecture Notes in Physics, **717**. Springer, Berlin, 2007.

[3] D. Bahns, A. Pohl, I. Witt, Ed.s, *Open quantum systems. A mathematical perspective.* Birkhäuser/Springer, 2019.

[4] B. Bertini, F. Heidrich-Meisner, C. Karrasch, T. Prosen, R. Steinigeweg, M. Žnidarič, *Finite-temperature transport in one-dimensional quantum lattice models*, Rev. Mod. Phys. **93** (2021), 025003 (1–71).

[5] L. Bertini, A. De Sole, G. Posta, *Trace distance ergodicity for quantum Markov semigroups*, J. Funct. Anal. **286** (2024).

[6] M. Brannan, L. Gao, M. Junge, *Complete logarithmic Sobolev inequalities via Ricci curvature bounded below*, Adv. Math. **394** (2022), 108129.

[7] H.P. Breuer, F. Petruccione, *The theory of open quantum systems.* Oxford University Press, New York, 2002.

[8] E.A. Carlen, J. Maas, *Gradient flow and entropy inequalities for quantum Markov semigroups with detailed balance.* Journal of Functional Analysis, **273** (2017), 1810–1869.

[9] E.A. Carlen, J. Maas, *Non-commutative calculus, optimal transport and functional inequalities in dissipative quantum systems.* J. Stat. Phys. **178** (2020), 319–378.

[10] E.A. Carlen, J. Maas, *Correction to: Non-commutative calculus, optimal transport and functional inequalities in dissipative quantum systems.* J. Stat. Phys. **181** (2020), 2432–2433.

[11] V. Gorini, A. Kossakowski, E. C. G. Sudarshan, *Completely positive dynamical semigroups of N-level systems.* J. Mathematical Phys. **17** (1976), 821–825.

[12] L. Gross, *Hypercontractivity and logarithmic Sobolev inequalities for the Clifford-Dirichlet form.* Duke Math. J. **42** (1975), 383–396.

[13] N. Datta, C. Rouzé, *Relating relative entropy, optimal transort and Fisher information: a quantum HWI inequality.* Ann. Henri Poincaré **21** (2020), 2115–2150.

[14] G. De Palma, M. Marvian, D. Trevisan, S. Lloyd, *The quantum Wasserstein distance of order 1.* IEEE Trans. Inform. Theory, **67** (2021), 6627–6643.

[15] G. De Palma, D. Trevisan, *The Wasserstein distance of order 1 for quantum spin systems on infinite lattices*, Ann. Henri Poincaré **24** (2023), 4237–4282.

[16] P. Naaijkens, *Quantum spin systems on infinite lattices.* Lecture Notes in Physics 933. Springer, Cham, 2017.

[17] P.-A. Meyer, *Quantum probability for probabilists.* Lecture Notes in Mathematics, 1538. Springer-Verlag, Berlin, 1993.

[18] T.M. Liggett, *Interacting particle systems.* Springer-Verlag, New York, 1985.

[19] G. Lindblad, *On the generators of quantum dynamical semigroups.* Comm. Math. Phys. **48** (1976), 119–130.

[20] V. Popkov, S. Essink, C. Presilla, and G. Schütz, *Effective quantum Zeno dynamics in dissipative quantum systems*, Phys. Rev. A **98** (2018), 052110 (1–8).

[21] M. Reed, B. Simon, *Method of modern mathematical physics II: Fourier analysis, self-adjointness.* Academic Press, London, 1975.

[22] D. Ruelle, *Statistical mechanics: rigorous results.* Addison-Wesley Publishing Co., Reading, Massachusetts 1969.

[23] M.M. Wilde *Quantum information theory.* Cambridge University Press 2016.

[24] M. Wirth, H. Zhang *Curvature-dimension conditions for symmetric quantum Markov semigroups.* Ann. Henri Poincaré, **24** (2023), 717—750.

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