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Letter

Bethe-Ansatz diagonalization of the steady state of boundary driven integrable spin chains

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Abstract
We find that the density operator of the nonequilibrium steady state (NESS) of XXZ spin chains with strong 'sink and source' boundary dissipation, can be described in terms of quasiparticles, with renormalized—dissipatively dressed—dispersion relation. The spectrum of the NESS is then fully accounted for by Bethe ansatz equations for an associated coherent system of these quasiparticles. The dissipative dressing generates an extra singularity in the dispersion relation, which significantly changes the NESS spectrum. In particular, it leads to a dissipation-assisted entropy reduction, due to the suppression

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in the NESS spectrum of plain wave-type Bethe states in favor of Bethe states localized at the boundaries.

Keywords: non-equilibrium steady state, dissipative systems, Bethe ansatz, XXZ spin chains, boundary driven integrable spin chains

1. Introduction

Integrable many-body systems play an important role in statistical mechanics. For instance, it was the celebrated Onsager solution of the 2D Ising model [1] that has proved, for the first time, that phase transitions are related to non-analyticity of the free energy functional, giving rise to the theory of critical phenomena. Solution of Korteweg–de Vries equation [2] has shown that a nonlinear system can possess an infinite number of local conservation laws, which led to a concept of nonlinear asymptotic Fourier transform [3] and to theory of stable nonlinear excitations—solitons. Integrable quantum many-body systems serve as reference points and as calibration tools for contemporary experiments in cold atoms [4, 5] and in quantum circuits [6].

One practical definition of integrability for isolated quantum many-body systems rests upon the possibility of finding their spectra with computational effort growing slower than exponentially in system size N. Indeed, finding specific solutions of the Bethe ansatz equations (BAEs) for an integrable Hamiltonian typically requires only poly(N) computation steps, while the dimension of the Hilbert space grows exponentially in N. This practical aspect of integrability is in full accordance with the theoretical treatment of integrable quantum Hamiltonians, which are centered around finding the BAEs for their spectra [7–9].

On the other hand, the situation with the same quantum systems in nonequilibrium, particularly for systems exposed to dissipation, is quite different. The central object for nonequilibrium systems is the nonequilibrium steady state (NESS) which is a (often unique) fixed point of time evolution. While, because of the discovery of the matrix product ansatz method [10, 11], the k-point correlations in the NESS can be calculated in a polynomial time, finding the NESS spectrum or even specific eigenvalues of nontrivial NESS has so far remained accessible by direct diagonalization only, i.e. still requiring exponentially (with systems size) growing effort.

Our aim is to present a method for finding the NESS spectrum of a famous quantum system, the XXZ spin chain, attached to boundary dissipation of the 'sink and source' type, via a Bethe Ansatz approach, that is, in polynomial time. This demonstrates a principal possibility of extending the above mentioned standard integrability paradigm to a certain family of integrable quantum systems in nonequilibrium. In this communication we present a mathematically rigorous proof for our chosen 'sink and source' scenario figure 1, while for other examples, where the validity of the same property is established numerically, we direct a reader to [12].

One remarkable feature of our NESS spectrum solution that should be mentioned beforehand, is related to a concept of quasiparticles. Quasiparticles are fundamental constituents of eigenstates of integrable coherent many-body systems: an energy E_α of any eigenstate $|\alpha\rangle$ is a sum of individual energies of all quasiparticles it contains: $E_\alpha = \sum_j \epsilon(u_{j\alpha})$ where $u_{j\alpha}$ is a set of quasiparticles rapidities and $\epsilon(u)$ is a dispersion relation.

What we find is that the NESS spectrum $\{\nu_\alpha\} = \{\exp(-\tilde{E}_\alpha)\}$ can be written in the same form $\tilde{E}_\alpha = \sum_j \tilde{\epsilon}(u_{j\alpha})$, where $u_{j\alpha}$ is a set of admissible rapidities for some related integrable coherent model, and $\tilde{\epsilon}(u)$ is a new effective dispersion relation characterizing 'coherent' quasiparticles 'renormalized' by the dissipation. Such a 'survival' of the intrinsically coherent objects – quasiparticles—under the action of dissipation, cannot be expected *a priori* and is very surprising.

2. XXZ model with sink and source

Our aim is to find the spectrum of NESS of a XXZ spin chain, with boundary dissipation, described by the Lindblad Master equation for the density matrix

$$\frac{\partial \boldsymbol{\rho}(\Gamma,t)}{\partial t} = -i\,[H,\boldsymbol{\rho}] + \Gamma\left(\mathcal{D}_{L_1}[\boldsymbol{\rho}] + \mathcal{D}_{L_2}[\boldsymbol{\rho}]\right) \tag{1}$$

$$H = \sum_{n=0}^{N}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\sigma_n^z\sigma_{n+1}^z\right), \tag{2}$$

$$L_1 = \sigma_0^-,\quad L_2 = \sigma_{N+1}^+, \tag{3}$$

$$\mathcal{D}_L[\boldsymbol{\rho}] = L\boldsymbol{\rho}L^\dagger - \frac{1}{2}\left(L^\dagger L\boldsymbol{\rho} + \boldsymbol{\rho}L^\dagger L\right). \tag{4}$$

With the choice (3) the Lindblad dissipators $\mathcal{D}_{L_1}$ and $\mathcal{D}_{L_2}$ make the edge spins (spins at sites 0 and $N+1$, see figure 1) relax towards targeted states $|\downarrow\rangle$, $|\uparrow\rangle$, with typical relaxation time $\tau_{\text{boundary}} = O(1/\Gamma)$. The matrix product Ansatz for NESS (but not for NESS spectrum!) is formulated in [13]. Here we shall consider the so-called quantum Zeno (QZ) regime, where τ_{boundary} is much smaller than typical time needed for the bulk relaxation $\tau_{\text{boundary}} \ll \tau_{\text{bulk}}$. Then, fast relaxation of the edge spins constrains the reduced density matrix of (1) to an approximately factorized form [14] $\boldsymbol{\rho}(\Gamma,t) = \rho_l \otimes \rho(\Gamma,t)\otimes\rho_r + O(1/\Gamma)$ for $t \gg 1/\Gamma$, where $\rho(\Gamma,t) \approx \mathrm{tr}_{0,N+1}\boldsymbol{\rho}(t)$ is the reduced density matrix of interior spins. We are interested in computing the spectrum $\{\nu_\alpha\}$ of the (unique) Zeno NESS,

$$\rho_{\text{NESS}} = \lim_{\Gamma\to\infty}\lim_{t\to\infty}\rho(\Gamma,t) = \sum_\alpha \nu_\alpha|\alpha\rangle\langle\alpha|. \tag{5}$$

To this end, we establish three crucial properties:

(i) ρ_{NESS} commutes with the Hamiltonian

$$[\rho_{\text{NESS}}, H_D] = 0, \tag{6}$$

$$H_D = \sum_{n=1}^{N-1}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\sigma_n^z\sigma_{n+1}^z\right) + \Delta\left(\sigma_N^z - \sigma_1^z\right), \tag{7}$$

(ii) ν_α are given by stationary solution of an auxiliary Markov process

$$\Gamma\frac{\mathrm{d}\nu_\alpha}{\mathrm{d}t} = \sum_{\beta\neq\alpha} w_{\beta\alpha}\nu_\beta - \nu_\alpha\sum_{\beta\neq\alpha} w_{\alpha\beta},\quad \alpha = 1,2,\ldots \tag{8}$$

with rates

$$w_{\beta\alpha} = |\langle\alpha|\sigma_1^-|\beta\rangle|^2 + |\langle\alpha|\sigma_N^+|\beta\rangle|^2, \tag{9}$$

where $|\alpha\rangle, |\beta\rangle$ are eigenstates of (7).

(iii) $\{\nu_\alpha\}$ satisfy the detailed balance condition

$$\nu_\alpha w_{\alpha\beta} = \nu_\beta w_{\beta\alpha}. \tag{10}$$

Let us comment on all the properties (i), (ii), (iii).

To obtain (i) we expand the time-independent steady state solution $\boldsymbol{\rho}(\Gamma)$ of (1) in powers of $1/\Gamma$, $\boldsymbol{\rho}(\Gamma) = \rho_0 + \Gamma^{-1}\rho_1 + \Gamma^{-2}\rho_2 + \dots$ Substituting into (1) we obtain the recurrence $i[H,\rho_k] = (\mathcal{D}_{L_1} + \mathcal{D}_{L_2})[\rho_{k+1}]$, for $k = 0, 1, \dots$ For the consistency of the recurrence $[H,\rho_k]$ must have no overlap with the kernel of the operator $(\mathcal{D}_{L_1} + \mathcal{D}_{L_2})$, equivalent to $\mathrm{tr}_{0,N+1}[H,\rho_k] = 0$. In the leading order $k = 0$, a substitution of $\rho_0 = \rho_l \otimes \rho_{\mathrm{NESS}} \otimes \rho_r$ into $\mathrm{tr}_{0,N+1}[H,\rho_0] = 0$ yields (6). The Hamiltonian (6) is called a dissipation-projected Hamiltonian [14] and can alternatively be obtained via a Dyson expansion.

To establish (ii) one writes down a Dyson expansion of the $\boldsymbol{\rho}(\Gamma,t)$ in (1) using $1/\Gamma$ as a perturbation parameter. This leads to an effective Lindblad evolution of the internal spins, see figure 1, of the form

$$\frac{\partial \rho(\Gamma,t)}{\partial t} = -i\left[H_D,\rho\right] + \frac{1}{\Gamma}\left(\mathcal{D}_{\sigma_1^-}\left[\rho\right] + \mathcal{D}_{\sigma_N^+}\left[\rho\right]\right) + O\left(\Gamma^{-2}\right), \tag{11}$$

containing a dominant coherent part with the Hamiltonian (7) and perturbative (of order $1/\Gamma$) Lindbladian dissipator. The latter gives rise to the auxiliary Markov process (8), (9). The procedure is explained in detail in [14, 15], and [12], so we shall omit it here.

Finally, the detailed balance (iii) equation (10) is a special property related to integrability of (7) and it is proved *a posteriori*, see appendix B.

Our strategy is to use the integrability of the Hamiltonian (7) in order to calculate the rates (9) and then the NESS spectrum via (10). Due to (6), ρ_{NESS} and H_D in (7) can be diagonalized simultaneously.

The Hamiltonian (7) is treated in its more general version (for arbitrary longitudinal boundary fields) in the pioneering paper of Sklyanin [16]. Due to $U(1)$ invariance, (7) can be block-diagonalized within blocks of fixed total magnetization $\sum_{n=1}^{N}\sigma_n^z$. The eigenvalues of the block with magnetization $N-2M$ are given by

$$E_\alpha = (N-1)\,\Delta + \sum_{j=1}^{M}\epsilon\left(u_{j,\alpha}\right), \tag{12}$$

$$\epsilon\left(u\right) = \frac{-2\sin^2\gamma}{\sinh\left(u+\frac{i\gamma}{2}\right)\sinh\left(u-\frac{i\gamma}{2}\right)}. \tag{13}$$

where $u_{j,\alpha}, j = 1,\dots,M$, are Bethe roots satisfying the set of BAE

$$\frac{u_j^{[-3]}}{u_j^{[+3]}}\left(\frac{u_j^{[+1]}}{u_j^{[-1]}}\right)^{2N+1} = \prod_{\substack{k=1\\k\neq j}}^{M}\prod_{\sigma=\pm1}\frac{\left(u_j+\sigma u_k\right)^{[+2]}}{\left(u_j+\sigma u_k\right)^{[-2]}}, \tag{14}$$

in which we defined $u^{[k]} = \sinh(u + ik\gamma/2)$, where $\cos\gamma = \Delta$.

Using the Algebraic Bethe Ansatz machinery and some previous results [17] we are able to calculate the rates $w_{\beta\alpha}$ from (9),(10) and hence the NESS spectrum $\{\nu_\alpha\}$. Details of the proof are given in appendix A. Usually, finding correlation functions for open integrable spin chains is a difficult task, and a result (if it exists) is expressed in terms of multiple integrals, infinite products, etc., see e.g. [17–20]. In our case, we succeeded to obtain an elegant explicit formula for the NESS spectrum, namely:

$$\nu_\alpha = A\exp\left(-\tilde{E}_\alpha\right),$$
$$\tilde{E}_\alpha = \sum_j \tilde{\epsilon}\left(u_{j,\alpha}\right) \tag{15}$$

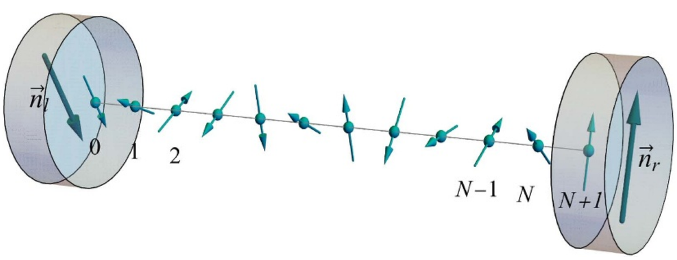

Figure 1. Schematic picture of the dissipative setup. The boundary spins are fixed by dissipation, while the internal spins follow an effective dynamics consisting of fast coherent dynamics (7) and slow relaxation dynamics (8) towards the NESS. For our 'sink and source' model (1) $\vec{n}_r = (0,0,1)$ and $\vec{n}_l = -\vec{n}_r$.

$$\begin{aligned} \tilde{\epsilon}(u) &= \log\left|\frac{\sinh\left(u+\frac{3i\gamma}{2}\right)\,\sinh\left(u-\frac{3i\gamma}{2}\right)}{\sinh\left(u+\frac{i\gamma}{2}\right)\,\sinh\left(u-\frac{i\gamma}{2}\right)}\right| \qquad 1 \\ &= \log\left|1-\epsilon(u)\Delta\right|, \qquad \Delta = \cos\gamma, \end{aligned} \tag{16}$$

where A is a normalization constant, and $\tilde{E}_\alpha, \tilde{\epsilon}(u)$ in (15), (16) are obvious dissipative analogs of the E_α, $\epsilon(u)$ in (12), (13).

To make the analogy even more obvious, we compare two reduced density matrices: a Gibbs state for the dissipation projected Hamiltonian (7) and the NESS, governed by (7):

$$\begin{aligned} \rho_{\text{Gibbs}} &= \frac{1}{Z}\sum_\alpha \mathrm{e}^{-\beta E_\alpha}|\alpha\rangle\langle\alpha|, \\ E_\alpha &= \sum_j \epsilon(u_{j\alpha}), \end{aligned} \tag{17}$$

$$\begin{aligned} \rho_{\text{NESS}} &= \frac{1}{\tilde{Z}}\sum_\alpha \mathrm{e}^{-\tilde{E}_\alpha}|\alpha\rangle\langle\alpha|, \\ \tilde{E}_\alpha &= \sum_j \tilde{\epsilon}(u_{j\alpha}). \end{aligned} \tag{18}$$

Note that both the eigenstates $|\alpha\rangle$ and the Bethe rapidities $u_{j,\alpha}$ (solutions of Bethe ansatz (14)) in (17) and (18) are the same.

It is instructive to compare singularities of the original dispersion $\epsilon(u)$ (13) and of the $\tilde{\epsilon}(u)$ (16). $\epsilon(u)$ has a singularity at $u_j = \pm i\gamma/2$. Its dissipative analog $\tilde{\epsilon}(u)$ (16) inherits the singularity $u_j = \pm i\gamma/2$ from $\epsilon(u)$ and 'dresses' $\epsilon(u)$ with an additional singularity at $u_j = \pm 3i\gamma/2$, see also figure 2. For this reason we shall call $\tilde{\epsilon}(u)$ a dissipatively dressed dispersion relation and $\tilde{E}_\alpha$ a dissipatively dressed energy. In other words, $\epsilon(u_j)$ is an energy of a quasiparticle with rapidity u_j and $\tilde{\epsilon}(u_j)$ the energy of the same quasiparticle in the dissipative setup (figure 1).

Equations (15) and (16) are our main results. We have shown that a spectrum of a NESS for the 'NESS-integrable' system (1) can be obtained by solving Bethe Ansatz equations for a related auxiliary coherent integrable problem (7). In [12] we argue that the effect of dissipative dressing of quasiparticles' dispersion is not restricted to the U(1)-symmetric XXZ model with sink and source (1) but it extends to other integrable spin chains, with $U(1)$ symmetry broken at the boundary and in the bulk. However, for the latter models a rigorous proof is currently lacking.

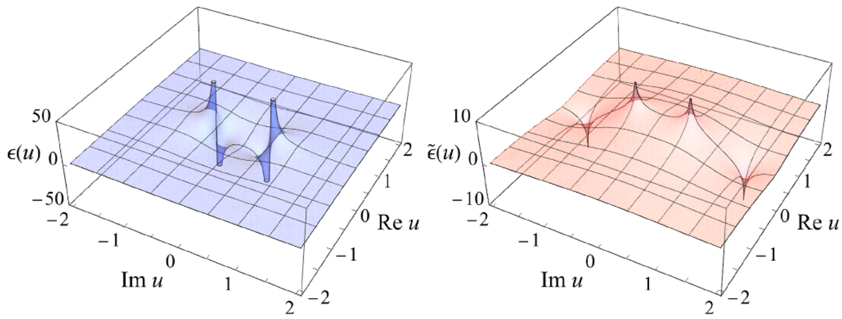

Figure 2. Surfaces $\epsilon(\mathrm{Re}\,u, \mathrm{Im}\,u)$ (left panel) and $\tilde{\epsilon}(\mathrm{Re}\,u, \mathrm{Im}\,u)$ (right panel) for isotropic case $\Delta = 1$ showing singularities at $u = \pm i/2$ and at $u = \pm i/2$, $u = \pm 3i/2$, respectively.

In the following we discuss some physical consequences of the dissipative dressing (16).

3. Dissipation assisted entropy reduction

Coupling of a quantum system to a dissipation typically leads to a mixed state and a volume-extensive entropy. In setups with quenches,leading to steady bulk currents, a volume-extensive entropy is also typical [21], as well as in mesoscopic conductors [22]. Here we show that subtle effect of dissipative dressing in the effective dispersion relation $\tilde{\epsilon}(u)$, in the contrary, can make the entropy subextensive in volume or even vanishingly small, see below.

To this end, we shall first discuss the isotropic case, $\Delta = 1$ in (7), which is obtained from the XXZ case by substituting $\gamma \to \delta$, $u \to \delta u$ and letting $\delta \to 0$. We then obtain BAE (14) with $u^{[q]} = u + iq/2$, while $\epsilon(u), \tilde{\epsilon}(u)$ are given by

$$\begin{aligned} \epsilon(u) &= -\frac{2}{u^2 + \frac{1}{4}}, \\ \tilde{\epsilon}(u) &= \log\left|\frac{u^2 + \frac{9}{4}}{u^2 + \frac{1}{4}}\right|. \end{aligned} \tag{19}$$

The surfaces $\epsilon(u), \tilde{\epsilon}(u)$ (19) for complex u are shown in figure 2. Let us consider one-particle sector $M = 1$, containing N Bethe eigenstates $|\alpha\rangle \equiv \psi(u_\alpha)$ parameterized by the solutions u_α, $\alpha = 1, \ldots N$, of the BAE (14) with $u^{[q]} = u + iq/2$. For any N the set $\{u_\alpha\}$ consists of $N-1$ real solutions, say $u_2, \ldots, u_N$ (corresponding to plane-wave type eigenstates) and one imaginary solution u_1 (corresponding to a boundary-localized eigenstates), see figure 3. The root u_1 lies exponentially close to the singularity $u = 3i/2$ of $\tilde{\epsilon}(u)$ in (19) due to dressing. Explicitly, from (14) we find $u_1 - 3i/2 \equiv \delta = 3i2^{-2N-1}(1 + O(N\delta))$, see [12] for a proof. The singularity of $\tilde{\epsilon}(u)$ drastically decreases $\tilde{\epsilon}(u_1)$ with respect to 'coherent' energy $\epsilon(u_1)$, the difference growing proportionally to the system size: $\tilde{\epsilon}(u_1) - \epsilon(u_1) = -2(N+1)\log 2 + O(1)$. On the other hand, for real (plain wave type) solutions, the dressed and original energies are comparable, see figure 3. As a result, the boundary localized Bethe eigenstate $\psi(u_1)$ enters the NESS (18) with an exponentially large relative weight with respect to the other eigenstates $\psi(u_2), \ldots, \psi(u_N)$ from the same one-quasiparticle sector $M = 1$, see figure 3.

The above mechanism predicting boundary-localized Bethe states to yield dominant contribution to the NESS (18) can be qualitatively extended to higher but fixed M. E.g. also in $M = 2, 3$ sectors there will be BAE roots exponentially close to the singularity $u = \pm 3i/2$

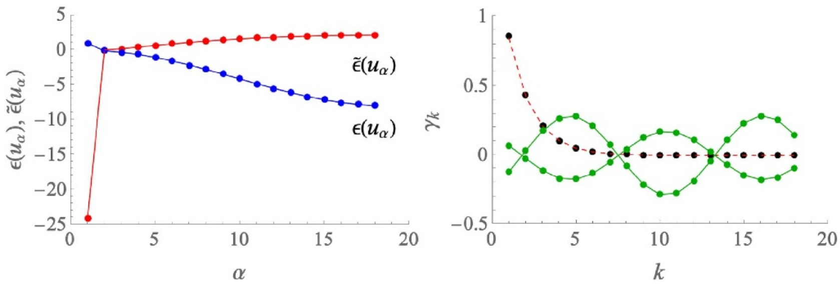

Figure 3. Left panel: quasiparticle energies $\epsilon(u_\alpha)$ (blue joined points) and $\tilde{\epsilon}(u_\alpha)$ (red joined points) in the XXX model with $N = 18$ spin, in the block with one magnon $M = 1$. The state $\alpha = 1$ is a localized Bethe state with $u_1 \simeq 3i/2 + i\,\mathrm{e}^{-24.5}$. Right panel: coefficients A_k of the normalized localized Bethe state $|\alpha = 1\rangle = \sum_{k=1}^{N} A_k\,\sigma_k^-\,|\uparrow\cdots\uparrow\rangle$ (black empty circles). The dashed red line is the fit $A_k = 1.7 \times 2^{-k}$. The green joined points are the coefficients $\mathrm{Re}A_k$ and $\mathrm{Im}A_k$ for the plain-wave like Bethe state with $u_4 \simeq 1.781\,39$.

of $\tilde{\epsilon}(u)$, corresponding to boundary-localized eigenstates of H_D. Understanding physics at fixed magnetization density M/N would require controlling thermodynamic Bethe Ansatz in the presence of boundary fields, which is beyond our present scope. However, we can see the net effect of dissipative dressing by looking at scaling of the von Neumann entropy $S(\rho) = -\mathrm{tr}(\rho \log \rho)$ with system size N. We observe the usual extensive volume law for Gibbs state (17), at $\beta = 1$, $S(\rho_{\mathrm{Gibbs}}) = O(N)$ and a clear sub-extensive scaling $S(\rho_{\mathrm{NESS}}) \simeq N^{0.3}$ for the NESS (18), see Left upper panel of figure 4. Keeping the dissipation term fixed, and breaking the integrability of (2) (by adding a staggered magnetic field), we ruin the subtle dressing mechanism, rendering both $S(\rho_{\mathrm{Gibbs}})$,$S(\rho_{\mathrm{NESS}})$ volume-extensive, which is a conventionally expected outcome, see left bottom panel of figure 4.

The difference of responses of an integrable and non-integrable spin chain to the boundary dissipation can be also well-seen on the distribution of eigenvalues $\{E_\alpha\}$ and $\{\tilde{E}_\alpha\}$, shown in figure 5.

The case $\Delta = 1$ considered above lies at the boundary between the easy plane gapless regime $|\Delta| < 1$ where correlations in XXZ model decay algebraically, and the easy axis gapped regime with correlations decaying exponentially with distance. The net effect of dressing (16) in the two cases is also different. Indeed, we observe distinct behavior of the entropy of NESS: a fast, perhaps exponential decay of $S(\rho_{\mathrm{NESS}})/N$ in the gapped regime $|\Delta| > 1$, and saturation to a finite value (restoring extensive volume law) $S(\rho_{\mathrm{NESS}})/N \sim \mathrm{const}$ in the gapless regime $|\Delta| < 1$, see two right Panels of figure 4. For $\Delta = 0$, see equation (16), we have $\tilde{\epsilon}(u) = 0$ leading to $\rho_{\mathrm{NESS}} = I$ and $S(\rho_{\mathrm{NESS}}) = N \log 2$. It is unclear at present if the extensive volume law $S(\rho_{\mathrm{NESS}}) \sim O(N)$ sets in for $|\Delta| < 1$ or some critical value of Δ_{crit} exists, above which the entropy is sublinear. Investigation of the one-particle sector $M = 1$, see [12] shows that a boundary-localized eigenstate for large N, exponentially close to the singularity $u = \pm 3i\gamma/2$ in (16) appears in the range $\frac{1}{2} < |\Delta| \leqslant 1$ allowing to propose the lower bound $\Delta_{\mathrm{crit}} \geqslant \frac{1}{2}$.

4. Discussion

We have developed an explicit Bethe Ansatz procedure for diagonalizing the steady state density operators of boundary dissipatively driven integrable quantum spin chains in the

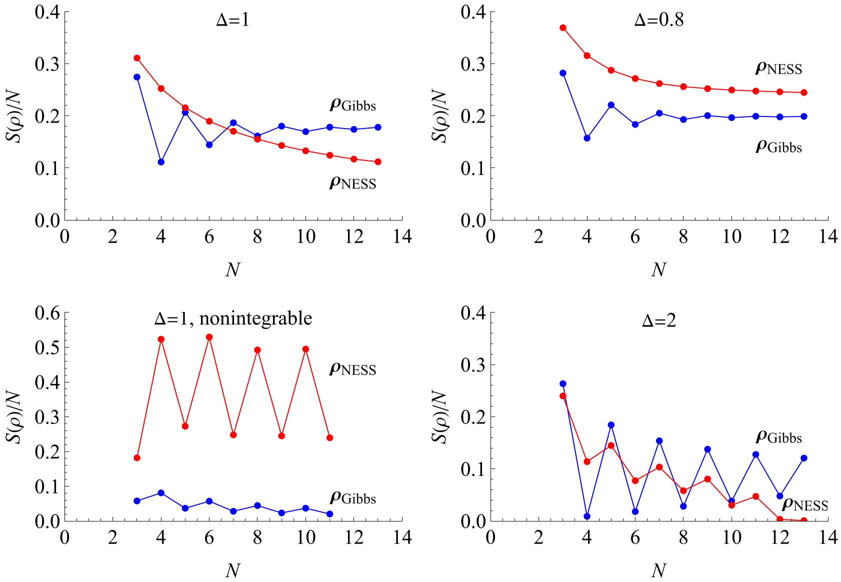

Figure 4. Von Neumann entropy $S(\rho) = -\mathrm{tr}(\rho \log \rho)$ per spin versus the system size N for $\rho = \rho_{\mathrm{Gibbs}}$ (blue points) and $\rho = \rho_{\mathrm{NESS}}$ (red points), illustrating the dissipation assisted entropy reduction. Left upper panel corresponds to the isotropic Heisenberg model $\Delta = 1$, red dashed line is a fit given by $2/(3N^{0.7})$. Left bottom panel shows a nonintegrable case in which a staggered magnetic field $(-1)^j h\sigma_j^z$, with $h = 1.5$, has been added. Right panels correspond to the anisotropic Heisenberg model in the easy plane $\Delta < 1$ and easy axis $\Delta > 1$ regime.

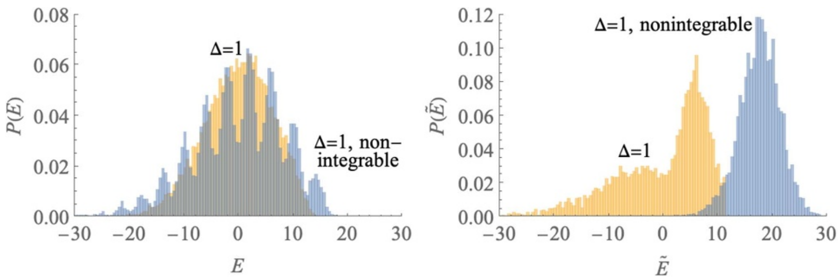

Figure 5. Distribution of the eigenvalues E of $-\log \rho_{\mathrm{Gibbs}}$ (left panel) and $\tilde{E}$ of $-\log \rho_{\mathrm{NESS}}$ (right panel) for the XXX model with $N = 13$. In both panels, the yellow and blue histograms correspond, respectively, to the integrable and nonintegrable cases of figure 4 with $\Delta = 1$. Distinct asymmetricity of $P(\tilde{E})$ (right Panel, yellow area) for the integrable case is due to the dissipative dressing effect.

limit of large dissipation, alias the Zeno regime. This becomes possible due to a surprising phenomenon of 'dissipative dressing' of quasiparticle energies in integrable coherent systems exposed to a dissipation. We find a general mechanism of entropy reduction due to dissipation—pushing the steady state density matrix towards a pure state—which is a consequence of additional singularities arising in the quasiparticle dispersion relation due to dissipative dressing.

Our results should have applications in state engineering and dissipative state preparation. Moreover, we expect analogous emergent integrability of the steady state in the discrete-time case of an integrable Floquet XXX/XXZ/XYZ circuit, where boundary dissipation can be conveniently implemented by the so-called reset channel [6].

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Derivation of equation (16)

We first recall some details of well-known Bethe Ansatz solution of an open XXZ model with diagonal boundary fields

$$H=\sum_{n=1}^{N-1}\left[\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y+\cosh\eta\,\sigma_n^z\sigma_{n+1}^z\right]+\frac{\sinh\eta}{\tanh p}\sigma_1^z+\frac{\sinh\eta}{\tanh q}\sigma_N^z. \tag{A.1}$$

Here, $\eta\equiv i\gamma$ is the anisotropic parameter. The boundary fields we are interested in are $-\cosh\eta\sigma_1^z$ and $\cosh\eta\sigma_N^z$ (see equation (7)), which corresponds to a substitution $q=-p=\eta$ in (A.1) . We shall perform the calculus for arbitrary p,q and make the substitution at the very end.

The R-matrix and K-matrices for the model are

$$R(u)=\begin{pmatrix}\sinh(u+\eta)&0&0&0\\0&\sinh u&\sinh\eta&0\\0&\sinh\eta&\sinh u&0\\0&0&0&\sinh(u+\eta)\end{pmatrix}, \tag{A.2}$$

$$K^-(u)=\begin{pmatrix}\sinh\left(p+u-\frac{\eta}{2}\right)&0\\0&\sinh\left(p-u+\frac{\eta}{2}\right)\end{pmatrix}, \tag{A.3}$$

$$K^+(u)=\begin{pmatrix}\sinh\left(q+u+\frac{\eta}{2}\right)&0\\0&\sinh\left(q-u-\frac{\eta}{2}\right)\end{pmatrix}. \tag{A.4}$$

Let us introduce the one-row monodromy matrices

$$\begin{aligned}T_0(u)&=R_{0,N}\left(u-\tfrac{\eta}{2}\right)\dots R_{0,1}\left(u-\tfrac{\eta}{2}\right),\\ \hat{T}_0(u)&=R_{1,0}\left(u-\tfrac{\eta}{2}\right)\dots R_{N,0}\left(u-\tfrac{\eta}{2}\right),\end{aligned} \tag{A.5}$$

and the double-row monodromy matrix

$$\mathcal{U}_0^-(u) = T_0(u) K_0^-(u) \hat{T}_0(u) = \begin{pmatrix} A_-(u) & B_-(u) \\ C_-(u) & D_-(u) \end{pmatrix}. \tag{A.6}$$

The transfer matrix is given by

$$t(u) = \mathrm{tr}_0 \left\{ K_0^+(u) \mathcal{U}_0^-(u) \right\}. \tag{A.7}$$

The Hamiltonian (A.1) rewritten in terms of the transfer matrix reads

$$H = \sinh\eta \left. \frac{\partial \ln t(u)}{\partial u} \right|_{u=\frac{\eta}{2}} - N\cosh\eta - \tanh\eta \sinh\eta. \tag{A.8}$$

The eigenvalue of the quantum transfer matrix can be given by the following $T-Q$ relation

$$\begin{aligned} \Lambda(u) = {} & \frac{\sinh(2u+\eta)}{\sinh(2u)} \prod_{s=p,q} \sinh\left(u+s-\tfrac{\eta}{2}\right) \sinh^{2N}\left(u+\tfrac{\eta}{2}\right) \\ & \times \prod_{j=1}^{m} \frac{\sinh(u-\lambda_j-\eta)\sinh(u+\lambda_j-\eta)}{\sinh(u-\lambda_j)\sinh(u+\lambda_j)} \\ & + \frac{\sinh(2u-\eta)}{\sinh(2u)} \prod_{s=p,q} \sinh\left(u-s+\tfrac{\eta}{2}\right) \sinh^{2N}\left(u-\tfrac{\eta}{2}\right) \\ & \times \prod_{j=1}^{m} \frac{\sinh(u+\lambda_j+\eta)\sinh(u-\lambda_j+\eta)}{\sinh(u+\lambda_j)\sinh(u-\lambda_j)}. \end{aligned} \tag{A.9}$$

The Bethe roots $\{\lambda_1,\ldots,\lambda_m\}$ in (A.9) satisfy the following BAE

$$\begin{aligned} & \left[\frac{\sinh\left(\lambda_j+\frac{\eta}{2}\right)}{\sinh\left(\lambda_j-\frac{\eta}{2}\right)}\right]^{2N} \prod_{s=p,q} \frac{\sinh\left(\lambda_j+s-\frac{\eta}{2}\right)}{\sinh\left(\lambda_j-s+\frac{\eta}{2}\right)} \\ & \times \prod_{k\neq j}^{m} \frac{\sinh(\lambda_j-\lambda_k-\eta)\sinh(\lambda_j+\lambda_k-\eta)}{\sinh(\lambda_j+\lambda_k+\eta)\sinh(\lambda_j-\lambda_k+\eta)} = 1, \qquad j=1,\ldots,m. \end{aligned} \tag{A.10}$$

Multiplying the left and right sides of BAE for all j, we get

$$\begin{aligned} & \prod_{j=1}^{m} \left[\frac{\sinh\left(\lambda_j+\frac{\eta}{2}\right)}{\sinh\left(\lambda_j-\frac{\eta}{2}\right)}\right]^{N} \frac{\sinh^{\frac{1}{2}}\left(\lambda_j+q-\frac{\eta}{2}\right)\sinh^{\frac{1}{2}}\left(\lambda_j+p-\frac{\eta}{2}\right)}{\sinh^{\frac{1}{2}}\left(\lambda_j-q+\frac{\eta}{2}\right)\sinh^{\frac{1}{2}}\left(\lambda_j-p+\frac{\eta}{2}\right)} \\ & \qquad = \pm \prod_{1\leqslant j<k\leqslant m} \frac{\sinh(\lambda_j+\lambda_k+\eta)}{\sinh(\lambda_j+\lambda_k-\eta)}. \end{aligned} \tag{A.11}$$

The energy of the system in terms of the Bethe roots reads

$$\begin{aligned} E = {} & \sum_{j=1}^{m} \frac{2\sinh^2\eta}{\sinh\left(\lambda_j+\frac{\eta}{2}\right)\sinh\left(\lambda_j-\frac{\eta}{2}\right)} + \sinh\eta(\coth p + \coth q) \\ & + (N-1)\cosh\eta. \end{aligned} \tag{A.12}$$

A.1. Expressing local operators $\sigma_1^{\pm}$ and $\sigma_N^{\pm}$ in terms of monodromy matrix elements

The trigonometric R-matrix possesses the following properties:

$$
\begin{aligned}
&R_{1,2}(0) = \sinh\eta P_{1,2}, \\
&R_{1,2}(u) R_{2,1}(-u) = \sinh(\eta+u)\sinh(\eta-u)\times\mathbb{I}, \\
&R_{1,2}(u) = -\sigma_1^y R_{1,2}^{t_1}(-u-\eta)\sigma_1^y.
\end{aligned}
\tag{A.13}
$$

It is easy to check that

$$
\operatorname{tr}_0\left\{\sigma_0^- K_0^+(u)\mathcal{U}_0^-(u)\right\} = \sinh\left(q+u+\tfrac{\eta}{2}\right)B_-(u), \tag{A.14}
$$

$$
\operatorname{tr}_0\left\{\sigma_0^+ K_0^+(u)\mathcal{U}_0^-(u)\right\} = \sinh\left(q-u-\tfrac{\eta}{2}\right)C_-(u). \tag{A.15}
$$

We have

$$
\begin{aligned}
&\operatorname{tr}_0\left\{\sigma_0^- K_0^+\left(-\tfrac{\eta}{2}\right)\mathcal{U}_0^-\left(-\tfrac{\eta}{2}\right)\right\} \\
&= \operatorname{tr}_0\left\{\left[\sigma_0^- K_0^+\left(-\tfrac{\eta}{2}\right)R_{0,N}(-\eta)\dots R_{0,1}(-\eta)\right]^{t_0}\left[K_0^-\left(-\tfrac{\eta}{2}\right)R_{1,0}(-\eta)\dots R_{N,0}(-\eta)\right]^{t_0}\right\} \\
&= \operatorname{tr}_0\left\{\left[R_{0,N}(-\eta)\dots R_{0,1}(-\eta)\right]^{t_0}\left[\sigma_0^- K_0^+\left(-\tfrac{\eta}{2}\right)\right]^{t_0}\left[R_{1,0}(-\eta)\dots R_{N,0}(-\eta)\right]^{t_0}\left[K_0^-\left(-\tfrac{\eta}{2}\right)\right]^{t_0}\right\} \\
&= \operatorname{tr}_0\left\{R_{0,1}^{t_0}(-\eta)\dots R_{0,N}^{t_0}(-\eta)\left[\sigma_0^- K_0^+\left(-\tfrac{\eta}{2}\right)\right]^{t_0}R_{N,0}^{t_0}(-\eta)\dots R_{1,0}^{t_0}(-\eta)\left[K_0^-\left(-\tfrac{\eta}{2}\right)\right]^{t_0}\right\} \\
&= \operatorname{tr}_0\left\{R_{0,1}(0)\dots R_{0,N}(0)\left[\sigma_0^y\sigma_0^- K_0^+\left(-\tfrac{\eta}{2}\right)\sigma_0^y\right]^{t_0}R_{N,0}(0)\dots R_{1,0}(0)\left[\sigma_0^y K_0^-\left(-\tfrac{\eta}{2}\right)\sigma_0^y\right]^{t_0}\right\} \\
&= \sinh^{2N}\eta\left[\sigma_N^y\sigma_N^- K_N^+\left(-\tfrac{\eta}{2}\right)\sigma_N^y\right]^{t_N}\operatorname{tr}_0\left\{\left[\sigma_0^y K_0^-\left(-\tfrac{\eta}{2}\right)\sigma_0^y\right]^{t_0}\right\} \\
&= -2\cosh\eta\sinh p\sinh q\sinh^{2N}\eta\,\sigma_N^-.
\end{aligned}
\tag{A.16}
$$

Analogously, we get

$$
\operatorname{tr}_0\left\{\sigma_0^+ K_0^+\left(-\tfrac{\eta}{2}\right)\mathcal{U}_0^-\left(-\tfrac{\eta}{2}\right)\right\} = -2\cosh\eta\sinh p\sinh q\sinh^{2N}\eta\,\sigma_N^+. \tag{A.17}
$$

Then, we get the following relation between $\sigma_N^{\pm}$ and $B_-(-\frac{\eta}{2})$, $C_-(-\frac{\eta}{2})$

$$
\sigma_N^- = -\frac{B_-\left(-\frac{\eta}{2}\right)}{2\cosh\eta\sinh p\sinh^{2N}\eta}, \tag{A.18}
$$

$$
\sigma_N^+ = -\frac{C_-\left(-\frac{\eta}{2}\right)}{2\cosh\eta\sinh p\sinh^{2N}\eta}. \tag{A.19}
$$

One can rewrite the transfer matrix in another way

$$
\begin{aligned}
t(u) &= \operatorname{tr}_0\left\{\left[K_0^+(u)T_0(u)\right]^{t_0}\left[K_0^-(u)\hat{T}_0(u)\right]^{t_0}\right\} \\
&= \operatorname{tr}_0\left\{T_0^{t_0}(u)\left[K_0^+(u)\right]^{t_0}\hat{T}_0^{t_0}(u)\left[K_0^-(u)\right]^{t_0}\right\} \\
&= \operatorname{tr}_0\left\{\left[K_0^-(u)\right]^{t_0}\left[\mathcal{U}_0^+(u)\right]^{t_0}\right\} \\
&= \operatorname{tr}_0\left\{K_0^-(u)\mathcal{U}_0^+(u)\right\},
\end{aligned}
\tag{A.20}
$$

where

$$\begin{aligned}\left[\mathcal{U}_0^+(u)\right]^{t_0} &= T_0^{t_0}(u)\left[K_0^+(u)\right]^{t_0}\hat{T}_0^{t_0}(u) \\ &= R_{0,1}^{t_0}\left(u-\tfrac{\eta}{2}\right)\dots R_{0,N}^{t_0}\left(u-\tfrac{\eta}{2}\right)\left[K_0^+(u)\right]^{t_0}R_{N,0}^{t_0}\left(u-\tfrac{\eta}{2}\right)\dots R_{1,0}^{t_0}\left(u-\tfrac{\eta}{2}\right) \\ &= \begin{pmatrix} A_+(u) & C_+(u) \\ B_+(u) & D_+(u)\end{pmatrix}. \end{aligned} \tag{A.21}$$

Using the same technique, we obtain that

$$\begin{aligned}\sinh p B_+\left(\tfrac{\eta}{2}\right) &= \mathrm{tr}_0\left\{\sigma_0^- K_0^-\left(\tfrac{\eta}{2}\right)\mathcal{U}_0^+\left(\tfrac{\eta}{2}\right)\right\} \\ &= \mathrm{tr}_0\left\{\left[\sigma_0^- K_0^-\left(\tfrac{\eta}{2}\right)\right]^{t_0}\left[\mathcal{U}_0^+\left(\tfrac{\eta}{2}\right)\right]^{t_0}\right\} \\ &= \mathrm{tr}_0\left\{\left[\sigma_0^- K_0^-\left(\tfrac{\eta}{2}\right)\right]^{t_0}T_0^{t_0}\left(\tfrac{\eta}{2}\right)\left[K_0^+\left(\tfrac{\eta}{2}\right)\right]^{t_0}\hat{T}_0^{t_0}\left(\tfrac{\eta}{2}\right)\right\} \\ &= \mathrm{tr}_0\left\{K_0^+\left(\tfrac{\eta}{2}\right)T_0\left(\tfrac{\eta}{2}\right)\sigma_0^- K_0^-\left(\tfrac{\eta}{2}\right)\hat{T}_0\left(\tfrac{\eta}{2}\right)\right\} \\ &= \mathrm{tr}_0\left\{K_0^+\left(\tfrac{\eta}{2}\right)R_{0,N}(0)\dots R_{1,0}(0)\,\sigma_0^- K_0^-\left(\tfrac{\eta}{2}\right)R_{1,0}(0)\dots R_{N,0}(0)\right\} \\ &= \sinh^{2N}\eta\,\sigma_1^- K_1^-\left(\tfrac{\eta}{2}\right)\mathrm{tr}_0\left\{K_0^+\left(\tfrac{\eta}{2}\right)\right\} \\ &= 2\cosh\eta\sinh q\sinh p\sinh^{2N}\eta\,\sigma_1^-, \end{aligned} \tag{A.22}$$

$$\sinh p\, C_+\left(\tfrac{\eta}{2}\right) = 2\cosh\eta\sinh q\sinh p\sinh^{2N}\eta\,\sigma_1^+. \tag{A.23}$$

To find the rates of the auxiliary Markov process, see the main text, we need to calculate the quantity

$$\frac{|\langle\approxeq|\sigma_1^-|\gtrapprox\rangle|^2+\langle\approxeq|\sigma_N^+|\gtrapprox\rangle|^2}{|\langle\gtrapprox|\sigma_1^-|\approxeq\rangle|^2+\langle\gtrapprox|\sigma_N^+|\approxeq\rangle|^2} = \frac{|\langle\approxeq|\sigma_1^-|\gtrapprox\rangle|^2}{|\langle\approxeq|\sigma_N^-|\gtrapprox\rangle|^2} = \left|\frac{\sinh p}{\sinh q}\right|^2\frac{|\langle\approxeq|B_+\left(\frac{\eta}{2}\right)|\gtrapprox\rangle|^2}{|\langle\approxeq|B_-\left(-\frac{\eta}{2}\right)|\gtrapprox\rangle|^2}, \tag{A.24}$$

where

$$\approxeq=\{u_1,\dots,u_{n+1}\}, \qquad \gtrapprox=\{v_1,\dots,v_n\}$$

are the solution of BAE (A.10) and $|\approxeq\rangle, |\gtrapprox\rangle$ are the corresponding Bethe state.

A.2. Calculating the expression (A.24)

In the following we will recall some result of [17] to derive the ratio

$$\frac{\langle\approxeq|B_+\left(\frac{\eta}{2}\right)|\gtrapprox\rangle}{\langle\approxeq|B_-\left(-\frac{\eta}{2}\right)|\gtrapprox\rangle}. \tag{A.25}$$

appearing in (A.24). The Bethe state $|\mathbf{u}\rangle$ can be constructed by either $\prod_{j=1}^{n+1}B_-(u_j)|\mathrm{vac}\rangle$ or $\prod_{j=1}^{n+1}B_+(u_j)|\mathrm{vac}\rangle$ where $|\mathrm{vac}\rangle = |\uparrow\cdots\uparrow\rangle$. Inserting the expression of $|\gtrapprox\rangle, |\approxeq\rangle$ into (A.25), we get

$$\begin{aligned}\frac{\langle\approxeq|B_+\left(\frac{\eta}{2}\right)|\gtrapprox\rangle}{\langle\approxeq|B_-\left(-\frac{\eta}{2}\right)|\gtrapprox\rangle} &= \frac{\langle\mathrm{vac}|C_-(u_1)\cdots C_-(u_{n+1})B_+\left(\frac{\eta}{2}\right)B_+(v_1)\dots B_+(v_n)|\mathrm{vac}\rangle}{\langle\mathrm{vac}|C_+(u_1)\cdots C_+(u_{n+1})B_-\left(-\frac{\eta}{2}\right)B_-(v_1)\dots B_-(v_n)|\mathrm{vac}\rangle} \\ &\quad\times\frac{\langle\mathrm{vac}|C_+(u_1)\cdots C_+(u_{n+1})\,B_-(v_1)\dots B_-(v_n)|\mathrm{vac}\rangle}{\langle\mathrm{vac}|C_-(u_1)\cdots C_-(u_{n+1})\,B_+(v_1)\dots B_+(v_n)|\mathrm{vac}\rangle}. \end{aligned} \tag{A.26}$$

From [17], we know that

$$
\begin{aligned}
&\frac{\langle \mathrm{vac}| C_+(u_1)\cdots C_+(u_{n+1})}{\langle \mathrm{vac}| C_-(u_1)\cdots C_-(u_{n+1})}\frac{B_-(v_1)\ldots B_-(v_n)|\mathrm{vac}\rangle}{B_+(v_1)\ldots B_+(v_n)|\mathrm{vac}\rangle} \\
&\quad=\prod_{j=1}^{n+1}\frac{\sinh(\eta-2u_j)}{\sinh(\eta+2u_j)}\prod_{j=1}^{n}\frac{\sinh(\eta+2v_j)}{\sinh(\eta-2v_j)}G(\{\succapprox\};q,p)\,G(\{\approxeq\};p,q)\,,
\end{aligned}
\tag{A.27}
$$

where

$$
\begin{aligned}
G(\{\lambda_1,\ldots,\lambda_m\};x,y)=&\prod_{j=1}^{m}\frac{\sinh^N\left(\lambda_j-\frac{\eta}{2}\right)}{\sinh^N\left(\lambda_j+\frac{\eta}{2}\right)}\frac{\sinh\left(\lambda_j-x+\frac{\eta}{2}\right)}{\sinh\left(\lambda_j+y-\frac{\eta}{2}\right)} \\
&\prod_{1\leqslant r<s\leqslant m}\frac{\sinh(\lambda_r+\lambda_s+\eta)}{\sinh(\lambda_r+\lambda_s-\eta)}.
\end{aligned}
\tag{A.28}
$$

With the help of theorem 4.1 and corollary 4.1 in [17], we get

$$
\begin{aligned}
&\frac{\langle \mathrm{vac}| C_-(u_1)\cdots C_-(u_{n+1})\,B_+\left(\frac{\eta}{2}\right)B_+(v_1)\ldots B_+(v_n)|\mathrm{vac}\rangle}{\langle \mathrm{vac}| C_+(u_1)\cdots C_+(u_{n+1})\,B_-\left(-\frac{\eta}{2}\right)B_-(v_1)\ldots B_-(v_n)|\mathrm{vac}\rangle} \\
&\quad=\frac{\langle \mathrm{vac}| C_-(u_1)\cdots C_-(u_{n+1})\,B_+\left(\frac{\eta}{2}\right)B_+(v_1)\ldots B_+(v_n)|\mathrm{vac}\rangle}{\langle \mathrm{vac}| C_+\left(\frac{\eta}{2}\right)C_+(-v_1)\cdots C_+(-v_n)\,B_-(-u_1)\ldots B_-(-u_{n+1})|\mathrm{vac}\rangle} \\
&\quad=\frac{\mathcal{S}^{-,+}_{n+1}\left(\{\approxeq\};\left\{\frac{\eta}{2},\succapprox\right\}\right)}{\mathcal{S}^{+,-}_{n+1}\left(\{-\approxeq\};\left\{\frac{\eta}{2},-\succapprox\right\}\right)},
\end{aligned}
\tag{A.29}
$$

where

$$
\mathcal{S}^{-,+}_{m}(\{\lambda\};\{\mu\})=\prod_{j=1}^{m}\sinh^N\left(\lambda_j-\tfrac{\eta}{2}\right)\sinh^N\left(\lambda_j+\tfrac{\eta}{2}\right)\frac{\det\mathcal{J}(\{\lambda\};\{\mu\})}{\det\mathcal{V}(\{\lambda\};\{\mu\})},
\tag{A.30}
$$

$$
\mathcal{S}^{+,-}_{m}(\{\mu\};\{\lambda\})=\prod_{j=1}^{m}\sinh^N\left(\lambda_j+\tfrac{\eta}{2}\right)\sinh^N\left(\lambda_j-\tfrac{\eta}{2}\right)\frac{\det\mathcal{J}(\{\lambda\};\{\mu\})}{\det\mathcal{V}(\{\lambda\};\{\mu\})},
\tag{A.31}
$$

$$
\mathcal{V}_{j,k}(\{\lambda\};\{\mu\})=\frac{\sinh(2\lambda_j)\sinh(2\mu_k-\eta)}{\sinh(2\lambda_j-\eta)\sinh(\mu_k-\lambda_j)\sinh(\mu_k+\lambda_j)},
\tag{A.32}
$$

$$
\mathcal{J}_{j,k}(\{\lambda\};\{\mu\})=\frac{\partial}{\partial\lambda_j}\Lambda(\mu_k,\{\lambda\})\,.
\tag{A.33}
$$

The identity

$$
\Lambda(u,\{\lambda\})=\Lambda(u,\{-\lambda\})=\Lambda(-u,\{-\lambda\})\,,
\tag{A.34}
$$

implies that

$$
\frac{\det\mathcal{J}\left(\{\approxeq\};\left\{\frac{\eta}{2},\succapprox\right\}\right)}{\det\mathcal{J}\left(\{-\approxeq\};\left\{\frac{\eta}{2},-\succapprox\right\}\right)}=(-1)^{n+1}\,.
\tag{A.35}
$$

For the matrix $\mathcal{V}$, we have

$$\frac{\mathcal{V}_{j,1}\left(\{\approxeq\};\{x,\succapprox\}\right)}{\mathcal{V}_{j,1}\left(\{-\approxeq\};\{x,-\succapprox\}\right)}=\frac{\sinh\left(2u_j+\eta\right)}{\sinh\left(2u_j-\eta\right)}, \tag{A.36}$$

$$\frac{\mathcal{V}_{j,k\neq 1}\left(\{\approxeq\};\{x,\succapprox\}\right)}{\mathcal{V}_{j,k\neq 1}\left(\{-\approxeq\};\{x,-\succapprox\}\right)}=-\frac{\sinh\left(2v_{k-1}-\eta\right)}{\sinh\left(2u_j-\eta\right)}\frac{\sinh\left(2u_j+\eta\right)}{\sinh\left(2v_{k-1}+\eta\right)}. \tag{A.37}$$

As a consequence, it can be verified that

$$\frac{\det\mathcal{V}\left(\{\approxeq\};\{\frac{\eta}{2},\succapprox\}\right)}{\det\mathcal{V}\left(\{-\approxeq\};\{\frac{\eta}{2},-\succapprox\}\right)}=(-1)^n\prod_{j=1}^{n+1}\frac{\sinh\left(2u_j+\eta\right)}{\sinh\left(2u_j-\eta\right)}\prod_{k=1}^{n}\frac{\sinh\left(2v_k-\eta\right)}{\sinh\left(2v_k+\eta\right)}. \tag{A.38}$$

Then, one can derive the following equation

$$\frac{\mathcal{S}^{-,+}_{n+1}\left(\{\approxeq\};\{\frac{\eta}{2},\succapprox\}\right)}{\mathcal{S}^{+,-}_{n+1}\left(\{-\approxeq\};\{\frac{\eta}{2},-\succapprox\}\right)}=-\prod_{j=1}^{n+1}\frac{\sinh\left(2u_j-\eta\right)}{\sinh\left(2u_j+\eta\right)}\prod_{k=1}^{n}\frac{\sinh\left(2v_k+\eta\right)}{\sinh\left(2v_k-\eta\right)}. \tag{A.39}$$

Substituting (A.27) and (A.39) into (A.26), we finally obtain

$$\begin{aligned}
\frac{\langle\approxeq|B_+\left(\frac{\eta}{2}\right)|\succapprox\rangle}{\langle\approxeq|B_-\left(-\frac{\eta}{2}\right)|\succapprox\rangle}&=G^{-1}\left(\{\succapprox\};q,p\right)G^{-1}\left(\{\approxeq\};p,q\right)\\
&=\prod_{j=1}^{n+1}\frac{\sinh^N\left(u_j+\frac{\eta}{2}\right)}{\sinh^N\left(u_j-\frac{\eta}{2}\right)}\prod_{j=1}^{n+1}\frac{\sinh\left(u_j+q-\frac{\eta}{2}\right)}{\sinh\left(u_j-p+\frac{\eta}{2}\right)}\\
&\quad\times\prod_{1\leqslant r<s\leqslant m}\frac{\sinh\left(u_r+u_s-\eta\right)}{\sinh\left(u_r+u_s+\eta\right)}\\
&\quad\times\prod_{k=1}^{n}\frac{\sinh^N\left(v_k+\frac{\eta}{2}\right)}{\sinh^N\left(v_k-\frac{\eta}{2}\right)}\frac{\sinh\left(v_k+p-\frac{\eta}{2}\right)}{\sinh\left(v_k-q+\frac{\eta}{2}\right)}\\
&\quad\times\prod_{1\leqslant r<s\leqslant n}\frac{\sinh\left(v_r+v_s-\eta\right)}{\sinh\left(v_r+v_s+\eta\right)}\\
&=\pm\prod_{j=1}^{n+1}\frac{\sinh^{\frac{1}{2}}\left(u_j+q-\frac{\eta}{2}\right)}{\sinh^{\frac{1}{2}}\left(u_j-p+\frac{\eta}{2}\right)}\frac{\sinh^{\frac{1}{2}}\left(u_j-q+\frac{\eta}{2}\right)}{\sinh^{\frac{1}{2}}\left(u_j+p-\frac{\eta}{2}\right)}\\
&\quad\times\prod_{k=1}^{n}\frac{\sinh^{\frac{1}{2}}\left(v_k+p-\frac{\eta}{2}\right)}{\sinh^{\frac{1}{2}}\left(v_k-q+\frac{\eta}{2}\right)}\frac{\sinh^{\frac{1}{2}}\left(v_k-p+\frac{\eta}{2}\right)}{\sinh^{\frac{1}{2}}\left(v_k+q-\frac{\eta}{2}\right)}.
\end{aligned} \tag{A.40}$$

When $q=-p=\eta$, the Bethe Ansatz equations are

$$\begin{aligned}
&\left[\frac{\sinh\left(\lambda_j+\frac{\eta}{2}\right)}{\sinh\left(\lambda_j-\frac{\eta}{2}\right)}\right]^{2N+1}\frac{\sinh\left(\lambda_j-\frac{3\eta}{2}\right)}{\sinh\left(\lambda_j+\frac{3\eta}{2}\right)}\\
&\quad\times\prod_{k\neq j}^{m}\frac{\sinh\left(\lambda_j-\lambda_k-\eta\right)\sinh\left(\lambda_j+\lambda_k-\eta\right)}{\sinh\left(\lambda_j+\lambda_k+\eta\right)\sinh\left(\lambda_j-\lambda_k+\eta\right)}=1,\qquad j=1,\ldots,m,
\end{aligned} \tag{A.41}$$

or,

$$\left[\frac{\sinh\left(\lambda_j+\frac{i\gamma}{2}\right)}{\sinh\left(\lambda_j-\frac{i\gamma}{2}\right)}\right]^{2N+1}\frac{\sinh\left(\lambda_j-\frac{3i\gamma}{2}\right)}{\sinh\left(\lambda_j+\frac{3i\gamma}{2}\right)}$$
$$\times\prod_{k\neq j}^{m}\frac{\sinh\left(\lambda_j-\lambda_k-i\gamma\right)\sinh\left(\lambda_j+\lambda_k-i\gamma\right)}{\sinh\left(\lambda_j+\lambda_k+i\gamma\right)\sinh\left(\lambda_j-\lambda_k+i\gamma\right)}=1,\qquad j=1,\ldots,m. \tag{A.42}$$

Then,

$$\frac{|\langle\approxeq|\sigma_1^-|\succapprox\rangle|^2+\langle\approxeq|\sigma_N^+|\succapprox\rangle|^2}{|\langle\succapprox|\sigma_1^-|\approxeq\rangle|^2+\langle\succapprox|\sigma_N^+|\approxeq\rangle|^2}$$
$$=\left|\prod_{j=1}^{n+1}\frac{\sinh\left(u_j+\frac{\eta}{2}\right)}{\sinh\left(u_j+\frac{3\eta}{2}\right)}\frac{\sinh\left(u_j-\frac{\eta}{2}\right)}{\sinh\left(u_j-\frac{3\eta}{2}\right)}\prod_{k=1}^{n}\frac{\sinh\left(v_k-\frac{3\eta}{2}\right)}{\sinh\left(v_k-\frac{\eta}{2}\right)}\frac{\sinh\left(v_k+\frac{3\eta}{2}\right)}{\sinh\left(v_k+\frac{\eta}{2}\right)}\right|$$
$$=\left|\prod_{j=1}^{n+1}\frac{\sinh\left(u_j+\frac{i\gamma}{2}\right)}{\sinh\left(u_j+\frac{3i\gamma}{2}\right)}\frac{\sinh\left(u_j-\frac{i\gamma}{2}\right)}{\sinh\left(u_j-\frac{3i\gamma}{2}\right)}\prod_{k=1}^{n}\frac{\sinh\left(v_k-\frac{3i\gamma}{2}\right)}{\sinh\left(v_k-\frac{i\gamma}{2}\right)}\frac{\sinh\left(v_k+\frac{3i\gamma}{2}\right)}{\sinh\left(v_k+\frac{i\gamma}{2}\right)}\right|, \tag{A.43}$$

leading to equation (16).

Appendix B. Kolmogorov property of the rates $w_{\alpha\beta}$. Proof of equation (10)

With the help of (A.43), we can derive that

$$w_{\alpha\beta}=\mathcal{F}_\alpha/\mathcal{F}_\beta, \tag{B.1}$$

where $\mathcal{F}_\alpha$ depends only on the Bethe roots, corresponding to eigenstate $|\alpha\rangle$. Therefore, we conclude that the following Kolmogorov relation holds,

$$w_{\alpha\beta_1}w_{\beta_1\gamma}w_{\gamma\beta_2}w_{\beta_2\alpha}=w_{\alpha\beta_2}w_{\beta_2\gamma}w_{\gamma\beta_1}w_{\beta_1\alpha}. \tag{B.2}$$

Equation (10) is a direct consequence of the Kolmogorov relation (B.2).

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