21.1. Derivation of the Path Integral

Let us assume that the Hamiltonian is time-independent and has the form

$$H=\frac{P^2}{2m}+V(X) \tag{21.1.1}$$

The propagator is defined by

$$U(xt;x'0)\equiv U(x,x',t)=\langle x|\exp\left(-\frac{i}{\hbar}Ht\right)|x'\rangle \tag{21.1.2}$$

It was stated in Chapter 8 that U may be written as a sum over paths going from $(x'0)$ to (xt). We will now see how this comes about.

First, it is evident that we may write

$$\exp\left(-\frac{i}{\hbar}Ht\right)=\left[\exp\left(-\frac{i}{\hbar}H\frac{t}{N}\right)\right]^N \tag{21.1.3}$$

for any N. This merely states that $U(t)$, the propagator for a time t, is the product of N propagators $U(t/N)$. Let us define

$$\varepsilon=\frac{t}{N} \tag{21.1.4}$$

and consider the limit $N\to\infty$. Now we can write

$$\exp\left(-\frac{i\varepsilon}{\hbar}(P^2/2m+V(X))\right)\simeq\exp\left(-\frac{i\varepsilon}{2m\hbar}P^2\right)\cdot\exp\left(-\frac{i\varepsilon}{\hbar}V(X)\right) \tag{21.1.5}$$

because of the fact that

$$e^Ae^B=e^{A+B+1/2[A,B]+\cdots} \tag{21.1.6}$$

which allows us to drop the commutator shown (and other higher-order nested commutators not shown) on the grounds that they are proportional to higher powers of ε which is going to 0. While all this is fine if A and B are finite dimensional matrices with finite matrix elements, it is clearly more delicate for operators in Hilbert space which could have large or even singular matrix elements. We will simply assume that in the limit $\varepsilon\to0$ the $\simeq$ sign in Eq. (21.1.5) will become the equality sign for the purpose of computing any reasonable physical quantity.

So we have to compute

$$\underbrace{\langle x|\exp\left(-\frac{i\varepsilon}{2m\hbar}P^2\right)\cdot\exp\left(-\frac{i\varepsilon}{\hbar}V(X)\right)\cdot\exp\left(-\frac{i\varepsilon}{2m\hbar}P^2\right)\cdot\exp\left(-\frac{i\varepsilon}{\hbar}V(X)\right)\ldots}_{N\text{ times}}|x'\rangle \tag{21.1.7}$$

The next step is to introduce the resolution of the identity:

$$I=\int_{-\infty}^{\infty}dx|x\rangle\langle x| \tag{21.1.8}$$

between every two adjacent factors of $U(t/N)$. Let us illustrate the outcome by considering $N=3$. We find (upon renaming x, x' as x_3, x_0 for reasons that will be clear soon)

$$\begin{aligned}U(x_3, x_0, t)=\int\prod_{n=1}^{2}dx_n\langle x_3|\exp\left(-\frac{i\varepsilon}{2m\hbar}P^2\right)\exp\left(-\frac{i\varepsilon}{\hbar}V(X)\right)|x_2\rangle\\ \times\langle x_2|\exp\left(-\frac{i\varepsilon}{2m\hbar}P^2\right)\exp\left(-\frac{i\varepsilon}{\hbar}V(X)\right)|x_1\rangle\\ \times\langle x_1|\exp\left(-\frac{i\varepsilon}{2m\hbar}P^2\right)\exp\left(-\frac{i\varepsilon}{\hbar}V(X)\right)|x_0\rangle\end{aligned} \tag{21.1.9}$$

Consider now the evaluation of the matrix element

$$\langle x_n|\exp\left(-\frac{i\varepsilon}{2m\hbar}P^2\right)\cdot\exp\left(-\frac{i\varepsilon}{\hbar}V(X)\right)|x_{n-1}\rangle \tag{21.1.10}$$

When the rightmost exponential operates on the ket to its right, the operator X gets replaced by the eigenvalue x_{n-1}. Thus,

$$\begin{aligned}&\langle x_n|\exp\left(-\frac{i\varepsilon}{2m\hbar}P^2\right)\cdot\exp\left(-\frac{i\varepsilon}{\hbar}V(X)\right)|x_{n-1}\rangle\\ &=\langle x_n|\exp\left(-\frac{i\varepsilon}{2m\hbar}P^2\right)|x_{n-1}\rangle\exp\left(-\frac{i\varepsilon}{\hbar}V(x_{n-1})\right)\end{aligned} \tag{21.1.11}$$

Consider now the remaining matrix element. It is simply the free particle propagator from x_{n-1} to x_n in time ε. We know what it is [say from Eq. (5.1.10)] or the following exercise

$$\langle x_n|\exp\left(\frac{-i\varepsilon}{2m\hbar}P^2\right)|x_{n-1}\rangle=\left[\frac{m}{2\pi i\hbar\varepsilon}\right]^{1/2}\exp\left[\frac{im(x_n-x_{n-1})^2}{2\hbar\varepsilon}\right] \tag{21.1.12}$$

Exercise 21.1.1. Derive the above result independently of Eq. (5.1.10) by introducing a resolution of the identity in terms of momentum states between the exponential operator and the position eigenket in the left-hand side of Eq. (21.1.12). That is, use

$$I=\int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}|p\rangle\langle p| \tag{21.1.13}$$

where the plane wave states have a wave function given by

$$\langle x|p\rangle=e^{ipx/\hbar} \tag{21.1.14}$$

which explains the measure for the p integration.

Resuming our derivation, we now have

$$\begin{aligned}\langle x_n|\exp\left(-\frac{i\varepsilon}{2m\hbar}P^2\right)\cdot\exp\left(-\frac{i\varepsilon}{\hbar}V(X)\right)|x_{n-1}\rangle\\=\left[\frac{m}{2\pi i\hbar\varepsilon}\right]^{1/2}\exp\left[\frac{im(x_n-x_{n-1})^2}{2\hbar\varepsilon}\right]\exp\left(-\frac{i\varepsilon}{\hbar}V(x_{n-1})\right)\end{aligned} \tag{21.1.15}$$

Collecting all such factors (there are just three in this case with $N=3$), we can readily see that for general N

$$\begin{aligned}U(x_N,x_0,t)=\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2}\left[\int\prod_{n=1}^{N-1}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2}dx_n\right]\\\times\exp\left[\sum_{n=1}^{N}\frac{im(x_n-x_{n-1})^2}{2\hbar\varepsilon}-\frac{i\varepsilon}{\hbar}V(x_{n-1})\right]\end{aligned} \tag{21.1.16}$$

If we drop the V terms we see that this is in exact agreement with the free particle path integral of Chapter 8. For example, the measure for integration has exactly N factors of B^{-1} as per Eq. (8.4.8), of which $N-1$ accompany the x-integrals. With the V term, the integrand is just the discretized version of $\exp(iS/\hbar)$:

$$\begin{aligned}\exp\left[\sum_{n=1}^{N}\frac{im(x_n-x_{n-1})^2}{2\hbar\varepsilon}-\frac{i\varepsilon}{\hbar}V(x_{n-1})\right]\\=\exp\frac{i}{\hbar}\varepsilon\sum_{n=1}^{N}\left[\frac{m(x_n-x_{n-1})^2}{2\varepsilon^2}-V(x_{n-1})\right]\end{aligned} \tag{21.1.17}$$

We can go back to the continuum notation and write all this as follows:

$$U(x, x', t) = \int [\mathscr{D}x] \exp\left[\frac{i}{\hbar}\int_0^t \mathscr{L}(x, \dot{x})\, dt\right] \tag{21.1.18}$$

where

$$\int [\mathscr{D}x] = \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2} \int \left[\prod_{n=1}^{N-1} \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2} dx_n\right] \tag{21.1.19}$$

The continuum notation is really a schematic for the discretized version that preceded it, and we need the latter to define what one means by the path integral. It is easy to make many mistakes if one forgets this. In particular, there is no reason to believe that replacing differences by derivatives is always legitimate. For example, in this problem, in a time ε, the variable being integrated over typically changes by $\mathcal{O}(\varepsilon^{1/2})$ and not $\mathcal{O}(\varepsilon)$, as explained in the discussion before Eq. (8.5.6). The works in the Bibliography at the end of this chapter discuss some of the subtleties. The continuum version is, however, very useful to bear in mind since it exposes some aspects of the theory that would not be so transparent otherwise. It is also very useful for getting the picture at the semiclassical level and for finding whatever connection there is between the macroscopic world of smooth paths and the quantum world. We will take up some examples later.

The path integral derived above is called the *Configuration Space* path integral or simply the path integral. We now consider another one. Let us go back to

$$\langle x_N | \underbrace{\exp\left(-\frac{i\varepsilon}{2m\hbar}P^2\right)\cdot \exp\left(-\frac{i\varepsilon}{\hbar}V(X)\right)\cdot \exp\left(-\frac{i\varepsilon}{2m\hbar}P^2\right)\cdot \exp\left(-\frac{i\varepsilon}{\hbar}V(X)\right)\ldots}_{N\text{ times}} |x_0\rangle \tag{21.1.20}$$

Let us now introduce resolutions of the identity between *every exponential* and the next. We need two versions

$$I = \int_{-\infty}^{\infty} dx |x\rangle\langle x| \tag{21.1.21}$$

$$I = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} |p\rangle\langle p| \tag{21.1.22}$$

where the plane wave states have a wave function given by

$$\langle x|p\rangle = e^{ipx/\hbar} \tag{21.1.23}$$

Let us first set $N=3$ and insert three resolutions of the identity in terms of p-states and two in terms of x-states with x and p resolutions alternating. This gives us

$$
\begin{aligned}
U(x_3, x_0, t) = \int [\mathscr{D}p\mathscr{D}x] \langle x_3 | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) | p_3\rangle \\
\times \langle p_3 | \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) | x_2\rangle \langle x_2 | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) | p_2\rangle \\
\times \langle p_2 | \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) | x_1\rangle \langle x_1 | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) | p_1\rangle \\
\times \langle p_1 | \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) | x_0\rangle
\end{aligned}
\tag{21.1.24}
$$

where

$$
\int [\mathscr{D}p\mathscr{D}x] = \underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{2N-1 \text{ times}} \prod_{n=1}^{N} \frac{dp_n}{2\pi\hbar} \prod_{n=1}^{N-1} dx_n \tag{21.1.25}
$$

Evaluating all the matrix elements of the exponential operators is trivial since each operator can act on the eigenstate to its right and get replaced by the eigenvalue. Collecting all the factors (a strongly recommended exercise for you) we obtain

$$
U(x, x', t) = \int [\mathscr{D}p\mathscr{D}x] \exp\left[\sum_{i=1}^{N} \left(\frac{-i\varepsilon}{2m\hbar} p_n^2 + \frac{i}{\hbar} p_n(x_n - x_{n-1}) - \frac{i\varepsilon}{\hbar} V(x_{n-1})\right)\right] \tag{21.1.26}
$$

This formula derived for $N=3$ is obviously true for any N. In the limit $N\to\infty$, i.e., $\varepsilon\to 0$, we write schematically in continuous time (upon multiplying and dividing the middle term by ε), the following continuum version:

$$
U(x, x'; t) = \int [\mathscr{D}p\mathscr{D}x] \exp\left[\frac{i}{\hbar}\int_0^t [p\dot{x} - \mathscr{H}(x, p)]\, dt\right] \tag{21.1.27}
$$

where $\mathscr{H} = p^2/2m + V(x)$ and $(x(t), p(t))$ are now written as functions of a continuous variable t. This is the *Phase Space Path Integral* for the propagator. The continuum version is very pretty [with the Lagrangian in the exponent, but expressed in terms of (x, p)] but is only a schematic for the discretized version preceding it.

In our problem, since p enters the Hamiltonian quadratically, it is possible to integrate out all the N variables p_n. Going back to the discretized form, we isolate the part that depends on just p's and do the integrals:

$$\prod_1^N \int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} \exp\left[\left(\frac{-i\varepsilon}{2m\hbar} p_n^2 + \frac{i}{\hbar} p_n(x_n - x_{n-1})\right)\right]$$

$$= \prod_1^N \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2} \exp\left[\frac{im(x_n - x_{n-1})^2}{2\hbar\varepsilon}\right] \tag{21.1.28}$$

If we now bring in the x-integrals we find that this gives us exactly the configuration space path integral, as it should.

Note that if p does not enter the Hamiltonian in a separable quadratic way, it will not be possible to integrate it out and get a path integral over just x, in that we do not know how to do non-Gaussian integrals. In that case we can only write down the phase space path integral.

We now turn to two applications that deal with the path integrals just discussed.

The Landau Levels

We now discuss a problem that is of great theoretical interest in the study of QHE (see Girvin and Prange). We now explore some aspects of it, not all having to do with functional integrals. Consider a particle of mass μ and charge q in the x–y plane with a uniform magnetic field B along the z-axis. This is a problem we discussed in Exercise (12.3.8). Using a vector potential

$$\mathbf{A} = \frac{B}{2}(-y\mathbf{i} + x\mathbf{j}) \tag{21.1.29}$$

we obtained a Hamiltonian

$$H = \frac{[P_x + qYB/2c]^2}{2\mu} + \frac{[P_y - qXB/2c]^2}{2\mu} \tag{21.1.30}$$

You were asked to verify that

$$Q = \frac{(cP_x + qYB/2)}{qB} \qquad P = (P_y - qBX/2c) \tag{21.1.31}$$

were canonical variables with $[Q, P] = i\hbar$. It followed that H was given by the formula

$$H = \frac{P^2}{2\mu} + \frac{1}{2}\mu\omega_0^2 Q^2 \tag{21.1.32}$$

and had a harmonic oscillator spectrum with spacing $\hbar\omega_0$, where

$$\omega_0 = qB/\mu c \tag{21.1.33}$$