Summary. — In this brief contribution we review the basic elements of self-assembly, calling attention on the competition between the energetic gain of forming a bond and the loss of translational entropy. We show how to calculate theoretically the distribution of cluster sizes, in the hypothesis of an ideal gas of cluster, and discuss how the cluster partition functions can be calculated numerically. Building on the thermodynamic formalism, we discuss some analytically soluble simple models of self-assembly: equilibrium and cooperative polymerization and micelle formation. Finally, we discuss the importance of directional interactions for self-assembly, its role in the bonding entropy reduction, in the suppression of the driving force for phase-separation and in the possibility of forming aggregates that do not expose attractive surfaces, thus minimizing inter-cluster attractive interactions.

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Basic concepts in self assembly

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The tendency of a system to spontaneously reach a well-defined structure is often named "self-assembly" [1]. Originally used in biology to describe the process that leads to the self organization of proteins into complex aggregates (like virus capsides), the word self-assembly has progressively permeated physics, chemistry and material science, becoming almost a substitute (despite its equivalence) for the more precise thermodynamic term, the minimization of the system free energy on going from a collection of disordered particles to the final (possibly ordered) structure. Crystallization of a metastable fluid can be properly considered as a well-known example of self-assembly.

While the search for a minimum free energy state is ubiquitous, the word self-assembly, especially in the context used in this school, is commonly limited to the case in which the final structure is composed by (often ordered) aggregates of finite size [2]: Micelles, vesicles, filaments, ribbons, self-assembling spontaneously under the appropriate external conditions (density, temperature, salt concentration, pH and so on).

Given the generality of the concept, a lecture on self-assembly can take an enormous number of directions (some of which are presented in details by the school lecturers). Here we will limit myself to some general consideration on self-assembly of one-component systems into finite size aggregates, and more specifically on its thermodynamic basis. Specifically, we will focus on particles whose interaction energy scale is indicated by ϵ and whose interaction range is indicated by Δ. We will use the generic word "particle", which according to the context can refer to an atom, a molecule, a macromolecule, a colloidal particle.

To exploit self-assembly into finite size structures it is fundamental to develop a thermodynamic description of the clustering process, highlighting the role of the bonding volume compared to the volume per particle and the bonding energy. The competition between the entropic driving force disfavouring self-assembly and the energetic (or enthalpic) driving force favouring the formation of low energy aggregates is at the heart of self-assembly. In the following, beside reviewing the thermodynamic approach, we will discuss some general aspects: (i) the need of operating at low k_BT (compared to ϵ); (ii) the need of directional attractive interactions (or alternatively of highly non-monotonic interaction potentials); (iii) the need to suppress collective phenomena (competition with crystallization and phase separation). Other important aspects of self assembly, such as how to predict the structure of the aggregate from the knowledge of the interaction potential (direct), how to design the interaction potential to spontaneously assemble a desired structure (inverse) and how external fields can be exploited to modify the routes toward the assembly of ordered structured are discussed in more detail in other chapters.

1. – Thermodynamic description of the clustering process

The first step in the description of an aggregation process is the definition of the aggregate (or cluster). A cluster is commonly defined as the set of all particles mutually connected (e.g. via a continuous path of bonds). Hence, the definition of a cluster as an entity requires preliminarily the definition of "bond" between particles. In most of the cases, bonding can be defined geometrically or energetically. For example, two particles can be considered bonded if they are closer than a bonding distance and with a proper relative orientation or if their pair interaction energy is smaller than a threshold. In the case of square-well like interactions, a clear-cut definition of bond is provided by the relative energy of the two particles: If such energy is the square-well depth $-\epsilon$ a bond exists. Thus, differently from extended infinite size aggregates — which can be stabilized also by repulsive interactions (as in the paradigmatic case of hard-sphere crystallization) — the formation of a stable aggregate of finite size requires attraction between the participating particles. For simplicity, we will assume generic interaction potentials to highlight the relative role of the interaction range (bonding volume) and of the interaction strength. The simplest case is the square-well potential. In this case a "bond" of strength $\epsilon > 0$ sets in when the distance between two particles is within σ and $\sigma + \Delta$ (being σ the characteristic particle size). The values of Δ and ϵ enter in the evaluation of the partition function of the system (and hence in the free energy) determining the temperature T and density ρ conditions under which bonding becomes statistically relevant. Typically, Δ is a fraction of σ (as it is discuss later on, larger Δ values do not allow for self-assembly of finite size clusters).

The thermodynamic description of an aggregating system significantly simplifies if the hypothesis that clusters do not significantly interact among themselves is satisfied (e.g. in the case in which the dominant contribution is excluded volume and the system packing fraction is small, so that clusters can be considered isolated). In this "ideal gas of cluster" approximation, the canonical partition function a system composed by clusters differing in the number n of bonded monomers, in the NVT ensemble (where N is the total number of monomers) can be written as [3]

$$Q = \prod_{n=1}^{\infty} \frac{Q_n^{N_n}}{N_n!} \tag{1}$$

where Q_n is the partition function of the n-cluster and N_n is the number of clusters of size n in the system. For spherical particles interacting with isotropic potentials

$$Q_n = \frac{1}{n!\lambda^{3n}} \int' d\mathbf{r_1}....d\mathbf{r_N} \exp\left[-\beta V(\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_N})\right] \tag{2}$$

where the $'$ sign in the integration limits indicates that only points in phase space $d\mathbf{r_1}....d\mathbf{r_N}$ for which the cluster does not break into disconnected smaller clusters should be considered.

For the monomer (assumed as a spherical rigid body, e.g. no internal fluctuations)

$$Q_1 = \frac{V}{\lambda^3}. \tag{3}$$

For directional interactions, one needs to integrate over all Euler angles of the particles Ω_j and the partition function becomes

$$Q_n = \frac{1}{n!\lambda^{3n}} \int' d\mathbf{r_1}....d\mathbf{r_N} d\Omega_1...d\Omega_n \exp\left[-\beta V(\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_N}, \Omega_1, ...\Omega_n)\right] \tag{4}$$

where now λ includes the rotational component of the integral over the kinetic energy. In these cases it is convenient to redefine $\Lambda^3 = \lambda^3 / \int d\Omega_1$ and define a spherically averaged partition function

$$Q_n = \frac{1}{n!\Lambda^{3n}} \frac{\int' d\mathbf{r_1}....d\mathbf{r_N} d\Omega_1...d\Omega_n \exp -\beta V(\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_N}, \Omega_1, ...\Omega_n)}{\int d\Omega_1...d\Omega_n} \tag{5}$$

The Helmholtz free energy F, the logarithm of the partition function, is then given by

$$\beta F = -\ln Q = -\sum_{n=1}^{\infty} \left(N_n \ln Q_n - N_n \ln N_n + N_n\right) = \sum_{n=1}^{\infty} N_n \left(\ln Q_n - \ln N_n + 1\right) \tag{6}$$

The cluster size distribution N_n is still undefined. To evaluate it, we require the free energy to be a minimum respect to all possible variation of N_n. Still, we must satisfy the constraint $\sum_n nN_n = N$. Introducing a Lagrange multiplier α to include the constraint we write

$$\frac{\partial(\beta F + \alpha \sum_k k N_k)}{\partial N_n} = 0 \tag{7}$$

which gives

$$\ln \frac{N_n}{Q_n} - n\alpha = 0 \tag{8}$$

or

$$N_n = Q_n (\exp \alpha)^n. \tag{9}$$

Since $N_1 = Q_1 \exp \alpha$, eq. (9) can be written as

$$N_n = Q_n \frac{N_1^n}{Q_1^n} = Q_n (\rho_1 \Lambda^3)^n \tag{10}$$

where ρ_1 is the monomer number density. This expression is particularly informative, since it shows that the probability of observing an aggregate of size n is proportional to the strength of the partition function Q_n, often defined as

$$Q_n = e^{-\beta f_n} \tag{11}$$

so that

$$\frac{N_n}{N} = Q_n \frac{N_1^n N^{n-1}}{N^n Q_1^n} = \left(\frac{N_1}{N}\right)^n \exp\left[-(\beta f_n + \ln N) + n(\beta f_1 + \ln N)\right] \tag{12}$$

Knowing the cluster size distribution is now possible to write a close expression for the system free energy in the ideal gas of cluster approximation as

$$\beta F = -\sum_{n=1}^{\infty} \left(N_n \ln Q_n - N_n \ln Q_n \frac{N_1^n}{Q_1^n} + N_n \right) = -\sum_{n=1}^{\infty} \left(n N_n \ln \frac{N_1}{Q_1} + N_n \right) = N \ln N_1/Q_1 - N_c \tag{13}$$

where N_c is the total number of clusters in the system. Considering that $\ln N_1/Q_1$ is the chemical potential of the monomer in the ideal gas approximation, βF can be written as

$$\beta F = N \beta \mu - N_c \tag{14}$$

It is interesting to note that, being an ideal gas, the pressure is proportional to the number of clusters ($N_c = \beta P V$) and that the monomer concentration (which fixes the value of μ) and the total number of clusters are the only pieces of information needed to write down the free energy.

2. – The need of low temperatures (compared to the bonding energy)

Before applying the previously derived formalism, it is useful to comment on the conditions which allow clusters to form for a very simple case in which we limit ourselves for simplicity to monomers and dimers. The formation of a dimer (e.g. the formation of a bond between two monomers) is proportional to Q_2. For a square-well interaction potential

$$Q_2 = \frac{1}{2!\Lambda^6} \int' d\mathbf{r_1} d\mathbf{r_2} \exp\left[-\beta V(\mathbf{r_1}, \mathbf{r_2})\right] = \frac{V}{2!\Lambda^6} \exp\left(\beta\epsilon\right) \frac{4\pi}{3} [(\sigma+\Delta)^3 - \sigma^3] \tag{15}$$

Defining V_b as the volume that allow for a bond (e.g. $V_b = \frac{4\pi}{3}[(\sigma+\Delta)^3 - \sigma^3]$)

$$N_2 = Q_n(\rho_1\Lambda^3)^2 = \frac{VV_b}{2}\frac{N_1^2}{V^2} \exp\left(\beta\epsilon\right) = N_1^2 \frac{V_b}{2V} \exp\left(\beta\epsilon\right) \tag{16}$$

The condition that half of the particles are dimers ($N_2 = N/4$ and $N_1 = N/2$) is reached when $\rho V_b \exp(\epsilon/k_B T) \approx 1$. This expression clearly show how the probability of forming bonds results from the competition of an entropic term (bonding volume compared to total volume per particle ρ^{-1}) and an energetic term (the Boltzmann factor). Assuming typical values for $\Delta \approx 0.1\sigma$ and for the packing fraction $\phi \equiv \frac{\pi}{6}\sigma^3\rho \approx 0.1$ this condition already teaches us that the probability that the bond is formed is not negligible only if $k_B T$ is smaller than ϵ ($k_B T/\epsilon \approx 0.2$). It is also important to consider the lifetime of the bonds, i.e. their persistence. As a first approximation, the lifetime of a bond is proportional to $\exp(\epsilon/k_B T)$. Thus at the typical temperature when fifty per cent of the particles are clustered [i.e. $\exp(\epsilon/k_B T) \approx 100$] the bonds are generally still very intermittent and the aggregates are to be considered as transient clusters with significant exchange of particles between aggregates. This immediately clarifies that to form a stable aggregate, i.e. an aggregate in which the relative position of the constituent particles is persistent in time, one need to go to $k_B T \ll \epsilon$. Requiring that $10^8 - 10^9$ attempts are needed before breaking a bond (this estimate of course depends on the attempt rate and the experimental observation time) implies $k_B T/\epsilon \approx 0.045 - 0.05$.

3. – The need of directional attractive interactions

The requirement of persistent bonds thus requires $k_B T$ to be significantly smaller than ϵ. At these low T, if the interaction potential is isotropic, the particles will have a tendency to phase separate, forming coarsening liquid droplets in the interior of which each particle is surrounded by approximately twelve neighbours. Self-assembly into finite size structure is thus pre-empted by phase separation.

Indeed, particles interacting via isotropic potentials (and with short-ranged interactions) are known to phase separate when the normalised second virial coefficient B_2^*

(normalized to the hard-sphere value B_2^{HS}) reaches the value $(B_2^*)_{T_c} \approx -1.2$ [4]. For the isotropic square-well potentials (our paradigmatic potential)

$$B_2^* \equiv \frac{B_2}{B_2^{HS}} = 1 - \frac{(\sigma+\Delta)^3 - \sigma^3}{\sigma^3} \left(\exp\left(\epsilon/k_B T\right) - 1\right) \tag{17}$$

which means that

$$\frac{k_B T_c}{\epsilon} = \left\{ \ln \left[1 + \frac{(1 - B_2^*)_{|T_c} \sigma^3}{[(\sigma+\Delta)^3 - \sigma^3]} \right] \right\}^{-1} \tag{18}$$

For a typical range of $\Delta \approx 0.1$, $k_B T_c/\epsilon \approx 0.5$ and hence it is impossible to bring the system to temperatures at which the bond lifetime would be sufficiently long to observe the presence of long-living well-defined aggregates without phase separation.

In principle one could attempt to decrease Δ to lower T_c. Still, eq. 18, in the limit of $\Delta/\sigma \to 0$, tends to

$$\frac{k_B T_c}{\epsilon} \approx \left\{ \ln \left[\frac{(1 - B_2^*{}_{|T_c})\sigma}{3\Delta} \right] \right\}^{-1} \approx (\ln \Delta/\sigma)^{-1} \tag{19}$$

which means that in the case of spherically interacting potentials only in the limit of very sticky (almost unphysical) interactions ($\Delta/\sigma \approx 10^{-9}$ or smaller) T_c is so small that the bond lifetime becomes longer than the experimental observation lifetime before phase separation is encountered, at least for interactions potentials which can be modelled as short-range attraction (^{1}).

Isotropic attractive interactions are thus not suited for self-assembly. Can the picture change with directional interactions? The answer is certainly positive. Directional attractive interactions offer many advantages. To retain simplicity, let us consider that the surface of the particle is decorated with patches and that a bond is present between the two particles when the relative distance between the particles is within $\sigma + \Delta$ (like in a square well interaction) but also when the orientation of the two patches involved in the bond is constrained within a specific value. In the commonly used Kern-Frenkel model [5], the center of each patch is described by a vector starting from the particle center. When the angle between the two vectors involved in the bonds is within a cone of semi-amplitude θ a bond is present. Compared to the square-well model, now the bonding volume is decreased by a factor χ^2 where $\chi = (1 - \cos\theta)/2$, commonly indicates the

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(^{1}) As a word of caution, we note that in soft matter it may happen than bonding arises as a result of an effective interaction which may involve a large entropic component. One typical case is offered by electrostatic interactions when counterions redistribute in space as a consequence of bonding or when significant conformational changes are associated to the bond-formation process (as in DNA self-assembly). Under these conditions, it is possible to modulate the bond lifetime and generate persistent bonds even for $T > T_c$.

fraction of solid angle accessible for bonding. For a conical bonding surface of semi-angle $\theta = 30°$ the bonding volume reduction is of the order of 10^3 compared to the isotropic case. This reduction in the entropy helps making the temperature at which bonds form and the temperature at which bonds are stable for sufficiently long times closer to each other.

There is another very important consequence of directional interactions: The possible suppression of the gas-liquid phase separation [6, 7]. Indeed, already the reduction in the number of bonded neighbours has a dramatic effect on the gas-liquid phase diagram, progressively decreasing both the critical temperature and the density of the liquid coexisting with the gas. This opens a region of packing fractions (above the coexisting liquid density but still smaller than the density at which packing becomes relevant) where it is possible to cool the system down to very low T without encountering phase separation.

The suppression of the gas-liquid phase separation is even more striking if the directional interactions favor the formation of aggregates that do not significantly attract each other. If bonding sites are completely saturated in each aggregate than these superparticles will feel each other essentially as hard-spheres, suppressing any driving force toward phase separation. This important consideration clarifies why long range attractive interactions and isotropic potentials are not suited for self-assembly. Fig. 1 shows schematic examples of one component system (Janus particles), interacting via hard-core (red) and square-well (green) potentials, which form aggregates similar to micelles and vesicles at low T for which the intra-aggregate attraction can be considered negligible [8, 9].

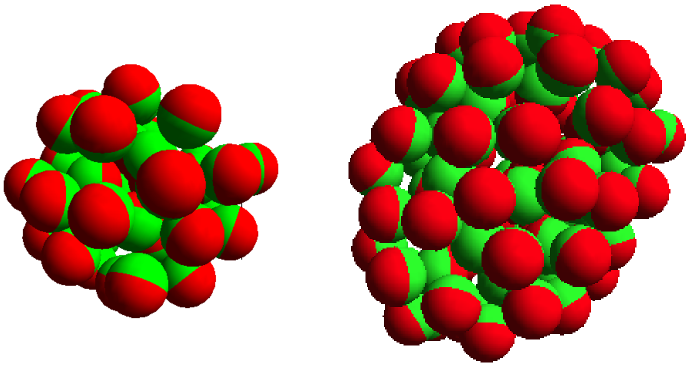

Fig. 1. – Schematic representation of a one patch Kern-Frenkel model with $\theta = \pi/2$, a simple model for a Janus particle. The two differently coloured areas indicate the attractive and repulsive parts of the particle surface. Particles self-assemble into clusters, with a preferential micelle (left) or vesicle (right) structure. In both geometries, the aggregate exposes to other clusters only a repulsive surface, preventing further aggregation and phase separation.

4. – Analogies with chemical reactions

Another way of describing clustering, equivalent to the previously derived thermodynamic formalism, can be developed in term of chemical reactions. Each cluster can be considered a different chemical species and the equilibration process that starts from a collection of monomers and ends into an equilibrium distribution of different clusters can be considered as the progression of the reactions toward equilibrium.

In this terminology, we could write

$$N_1 + N_1 \Longleftrightarrow N_2 \tag{20}$$

$$N_1 + N_2 \Longleftrightarrow N_3 \tag{21}$$

and so on and associate to each of these reactions a reaction constant K_n (with the dimension of inverse concentration). We would thus write

$$\frac{[N_2]}{[N_1]^2} = K_2 \tag{22}$$

$$\frac{[N_3]}{[N_1][N_2]} = K_3 \tag{23}$$

or substituting the precedent expression

$$\frac{[N_3]}{[N_1]^3} = K_2 K_3 \tag{24}$$

and so on. To grasp a feeling of the meaning of the value of K, we note that in the case of an isolated chemical reaction [e.g. eq. (22)] the equilibrium constant indicates the volume per particle at which half of the particles are in monomeric state and half in dimeric. Indeed, due to particle conservation, when $N_1 = N/2$, $N_2 = N_1/2$ and

$$K_2 = \frac{N/4V}{(N/2)^2} = \frac{V}{N}|_{N_1=\frac{N}{2}} \tag{25}$$

Hence, K [which has the dimension of a volume, for reactions as the one reported in eq. (22] indicates the volume per particle at which the reaction has progressed fifty per cent.

With the expression previously derived for the cluster size distribution in the ideal gas of cluster approximation [eq. (10)]

$$\frac{[N_2]}{[N_1]^2} = K_2 = \frac{Q_2}{Q_1^2 V} \tag{26}$$

$$\frac{[N_3]}{[N_1]^3} = K_2 K_3 = \frac{Q_3}{Q_1^3 V^2} \tag{27}$$

or $K_3 = Q_3/(Q_1 Q_2 V)$

Chemical constants provide information on the change in free energy associated to the clustering process at fixed center of mass, e.g. independently from the system volume. For example, for the reaction

$$N_1 + N_1 \Longleftrightarrow N_2 \tag{28}$$

K_2 is given by

$$K_2 = \frac{N_2/V}{(N_1/V)^2} = V\frac{N_2}{N_1^2} = V\frac{Q_2 N_1^2 \Lambda^6}{N_1^2} = V\frac{V K_{bond}}{\Lambda^6}\Lambda^6 = K_{bond} \tag{29}$$

where

$$K_{bond} = \int' d\mathbf{r_{12}} \exp\left[-\beta V(\mathbf{r_{12}})\right] \approx V_{bond} \exp\left(\beta\epsilon\right). \tag{30}$$

5. – The simplest self-assembly process. Equilibrium polymerization

The simplest case of self-assembly refers to particles that can form two bonds each (e.g. particles with functionality $f = 2$) [10, 11, 12, 13]. To evaluate the partition function we assume that particles interact with the Kern-Frenkel [5] model. We consider that the surface of the particle is decorated with two patches on the poles and that a bond is present between the two patches when the relative distance between the particles is within $\sigma + \Delta$ and when the orientation of both patches involved in the bonds is within a cone of half-angle θ. For example, for a dimer we have

$$Q_2 = \frac{1}{2!\Lambda^6}\int' d\mathbf{r_1} d\mathbf{r_2} d\Omega_1 d\Omega_2 e^{-\beta V(\mathbf{r_1},\mathbf{r_2},\Omega_1,\Omega_2)} / \int d\Omega_1 d\Omega_2 \tag{31}$$

With the simple model selected, the Boltzmann factor $\exp\left(\beta\epsilon\right)$ is constant in all points in space where a bond is present. Changing variable to $\mathbf{r_1}$ and $\mathbf{r_2} - \mathbf{r_1}$, the integration over $\mathbf{r_1}$ is immediate and results in a V term. The integration over $\mathbf{r_2} - \mathbf{r_1}$ is limited for relative distances between σ and $\sigma + \Delta$ and so it gives $\frac{4}{3}\pi[(\sigma + \Delta)^3 - \sigma^3)]$ times the integration over the bonding angles. Normalized by the factor of $(4\pi)^2$, the angular part results in a contribution $(1 - \cos\theta)^2/4$. This last term correspond to the so-called coverage χ (the fraction of the sphere surface associated to bonding) squared. The resulting partition function is thus

$$Q_2 = \frac{V}{2\Lambda^6} f^2 \frac{4}{3}\pi[(\sigma + \Delta)^3 - \sigma^3)]\chi^2 \exp\left(\beta\epsilon\right) \tag{32}$$

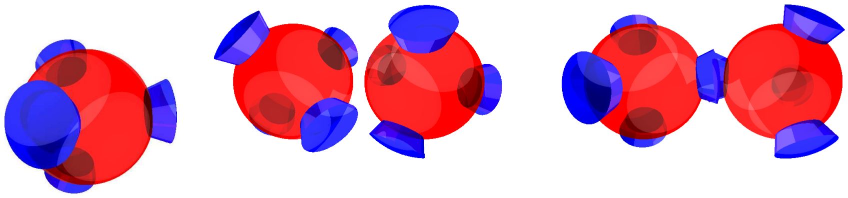

Fig. 2. – Pictorial representation of a Kern-Frenkel particle with four patches (left), in a non-bonded dimer configuration (center) and in a bonded one (right)

where the term f^2 counts the four ways a bond can be formed between two particles with two patches each. Using the previously introduced bonding volume definition,

$$Q_2 = 2\frac{VV_b}{\Lambda^6}\exp{(\beta\epsilon)} = \frac{V}{\Lambda^3}Q_{bond} \qquad Q_{bond} = 2\frac{V_b}{\Lambda^3}\exp{(\beta\epsilon)} \tag{33}$$

where the term V/λ^3 indicates the contribution to the partition function associated to the exploration of the system volume of the cluster center of mass, while the remaining part is the bond partition function in which the term $2V_b/\lambda^3$ counts the number of microstates associated to the existence of the bond and $\exp{(\beta\epsilon)}$ is the Boltzmann term, which depends on the ratio between the bond energy and the thermal energy.

Generalization to the case of a cluster of size n (neglecting self-avoiding contributions and under the assumption that there is no change in the bonding energy on clustering, the so-called isodesmic hypothesis, implicit in the simple classical potential that we are using) the partition function can be written as

$$Q_n^{f=2} = \frac{\omega_n}{n!\lambda^{3n}}V[V_b^{11}\exp{(\beta\epsilon)}]^{N_b} \tag{34}$$

with the number of bonds $N_b = n - 1$ and

$$\frac{\omega_n}{n!} = 2^{n-1} \tag{35}$$

where ω_n counts the number of distinct bonded chains that can be formed by n distinguishable particles. To calculate ω_n one considers that the first particle can be selected in n ways and that it has two possible bonding configurations. The second one among the $n-1$ remaining particles, always with two bonding possibilities. Hence

$$\omega_n = 2n \times 2(n-1) \times 2(n-2) \times ..... \times 2 = n!\ 2^{n-1} \tag{36}$$

and

$$Q_n^{f=2} = 2^{n-1} \frac{V}{\lambda^3} \left[\frac{V_b}{\lambda^3} \exp(\beta\epsilon) \right]^{n-1} = \frac{V}{\lambda^3} \left[2\frac{V_b}{\lambda^3} \exp(\beta\epsilon) \right]^{n-1} = \frac{V}{\Lambda^3} Q_{bond}^{n-1} \tag{37}$$

which can be interpreted as the center of mass partition function (V/Λ^3) and the bonds ($n-1$) partition function Q_{bond}^{n-1}.

The cluster size distribution is then given by

$$\begin{aligned} N_n = \frac{N_1^n}{Q_1^n} Q_n = \left(\frac{N_1 \lambda^3}{V} \right)^n \frac{V}{\lambda^3} Q_{bond}^{n-1} = \rho_1 V \left(\rho_1 \lambda^3 Q_{bond} \right)^{n-1} = \\ = N_1 \left(\rho_1 \lambda^3 Q_{bond} \right)^{n-1} = N_1 e^{(n-1)\ln(\rho_1 \lambda^3 Q_{bond})} \end{aligned} \tag{38}$$

e.g. an exponential distribution of polymer lengths, with characteristic decay $\bar{n} = -[\ln(\rho_1 \lambda^3 Q_{bond})]^{-1}$.

6. – Equilibrium polimerization in chemical language

In chemical language, the case of equilibrium polymerization correspond to assuming that the equilibrium constant K_n are all identical and equal to K_2. Under this hypothesis,

$$\frac{[N_n]}{[N_1]^n} = K_2 K_3 .... K_n = \prod_2^n K_n = K_2^{n-1} \tag{39}$$

and

$$N_n = N_1^n (K_2/V)^{n-1} = N_1 (K_2 \rho_1)^{n-1} \tag{40}$$

and remembering that $K_2 = Q_2/(Q_1^2 V)$ we recover the same expression we derived thermodynamically in the previous section

$$N_n = N_1 \left(Q_b \rho_1 \lambda^3 \right)^{n-1} . \tag{41}$$

7. – Cooperative polymerization: Slaved equilibrium polymerization

A relevant case of self-assembly is provided by the "explosive" formation of very long one dimensional aggregates (fibers, fibrils and so on) [14, 15]. In this cases, a very small change in the external control parameters determines the formation of extremely long chains. This fast growth of fibers originates from the presence of two distinct aggregation mechanisms. A very slow preliminary aggregation process, with a very small reaction constant and a subsequent fast aggregation process with a large reaction constant. A typical example is provided by the coil to helix transition, where first four monomers need to arrange in a proto-helix configuration and then the helix polymerization is rather fast.

In chemical language cooperative polymerization is described (in its simplest form) by the expressions

$$\frac{[N_2]}{[N_1]^2} = K_2 \tag{42}$$

$$\frac{[N_3]}{[N_1][N_2]} = K_3 \tag{43}$$

while all successive K_n terms are equal to K_3. In other words, we assume that first one need to nucleate a dimer and then the dimer can grow with an isodesmic process. In this case

$$\frac{[N_n]}{[N_{n-1}][N_1]} = K_3, \qquad n \geq 3 \tag{44}$$

In terms of concentrations

$$\frac{[N_2]}{[N_1]^2} = K_2 \tag{45}$$

and

$$\frac{[N_n]}{[N_1]^n} = K_2 K_3^{n-2}, \qquad n \geq 3 \tag{46}$$

The total monomer concentration can thus be written as

$$\rho = \sum_1^\infty n N_n = N_1 + K_2 N_1^2 + \sum_3^\infty K_2 K_3^{n-2} [N_1]^n = \tag{47}$$

$$= [N_1] + 2K_2[N_1]^2 + \sum_3^\infty n K_2 K_3^{n-2}[N_1]^n = [N_1] + \sum_2^\infty n K_2 K_3^{n-2} [N_1]^n$$

and using

$$\sum_2^\infty n x^n = \frac{(2-x)x^2}{1-x)^2} \tag{48}$$

we find that

$$\rho = [N_1] + \frac{K_2}{K_3} \frac{(2 - K_3[N_1])K_3[N_1]^2}{(1 - K_3[N_1])^2} \tag{49}$$

By multiplying by K_3 one obtains a dimensionless expression

$$K_3\rho = K_3[N_1] + \frac{K_2}{K_3} \frac{(2 - K_3[N_1])K_3^2[N_1]^2}{(1 - K_3[N_1])^2} \tag{50}$$

Figure 3 shows that for small K_2/K_3 the total density coincides with the monomer density till $\rho = K_3^{-1}$ and then it abruptly decay to zero.

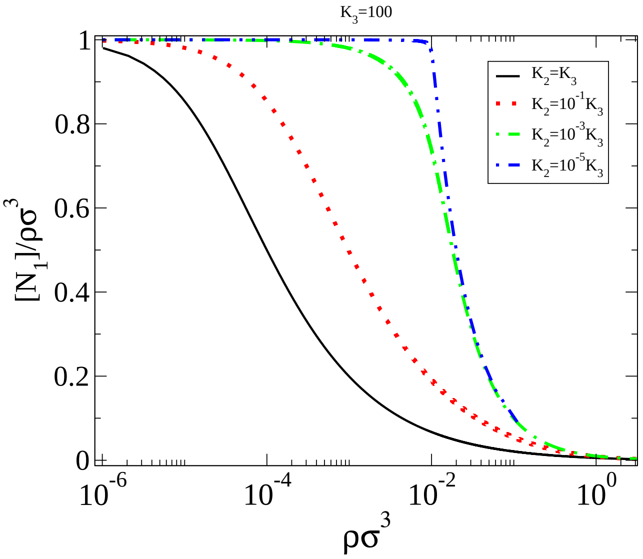

Fig. 3. – Plot of the fraction of particles in monomeric state $[N_1]/\rho$ as a function of the total density ρ for different values of the ratio K_2/K_3. Note the abrupt onset of polymerisation when K_2/K_3 is small. Here $K_3 = 100\sigma^3$

Recently, a simple model for patchy particles interacting with pair-wise additive interactions has been shown to undergoes cooperative polymerisation [16], forming abruptly extremely long tubes, as shown in fig. 4

8. – Micelles

In this section we present a minimal model for micelle aggregation [2, 17]. For the sake of simplicity we assume that particles can only exist in monomeric state or in a cluster of $M \gg 1$ particles (the micelle).

As we have demonstrated previously, in the case of an ideal gas of non-interacting clusters

$$N_M = Q_M \frac{N_1^M}{Q_1^M} \tag{51}$$

Then, denoting by N the original number of particles in the system

$$N = N_1 + M N_M = N_1 + M Q_M \frac{N_1^M}{Q_1^M} \tag{52}$$

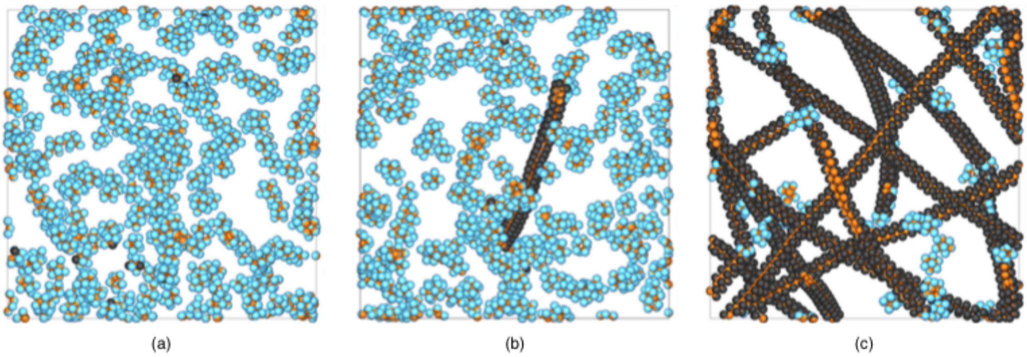

Fig. 4. – Images from a simulation of Kern-Frenkel particles with one attractive patch at three different (but very close) temperatures. Before the transition (left) the system is composed by finite size clusters, while after the transition most of the particles are part of very long and persistent polymers. Redrawn from Ref. [16].

or

$$\frac{N_1}{N} = 1 - MQ_M \frac{N_1^M}{NQ_1^M} \tag{53}$$

Now we can write, assuming that a micelle has a well defined energy E_M in all of its configurations, the partition function of the micelle as

$$Q_M = \frac{V}{\lambda^3} \left(\frac{V_b}{\lambda^3}\right)^{M-1} \exp(-\beta E_M) \tag{54}$$

to emphasize the entropic and energetic (or enthalpic) contributions and the partition function of the monomer $Q_1 = V/\lambda^3$. Then

$$\frac{N_1}{N} = 1 - M\frac{V}{\lambda^3} \left(\frac{V_b}{\lambda^3}\right)^{M-1} \exp(-\beta E_M) \frac{N_1^M}{N(\frac{V}{\lambda^3})^M} \tag{55}$$

and after some algebra

$$\frac{N_1}{N} = 1 - M \left(\frac{N_1}{N}\right)^M \left(\frac{NV_b}{V}\right)^{M-1} \exp(-\beta E_M) \tag{56}$$

which can be written symbolically as $x = 1 - x^M A$, with $A = M \left(\frac{NV_b}{V}\right)^{M-1} \exp(-\beta E_M)$. Fixing the properties of the micelle (M and the model parameters V_b and E_M) it is possible to solve eq. (56) for all densities and temperatures. The solution for N_1/N depends on the value of A. For values of A smaller than one, $N_1/N \approx 1$ and the system is a monomeric state. For A greater than N, $\rho_1 \equiv N_1/V$ reaches a constant value, as shown in Fig. **8**. Note also that both the entropic and the energetic contributions scale with M. Hence, the cross-over from values smaller than one to values larger than one is extremely fast. The concentration for which $A = 1$ is commonly referred to as critical micelle concentration (cmc). Figure 5 in ref. [9] shows the analog of Fig. **8** for the case of Janus colloidal particles aggregating in micelles and vesicles.

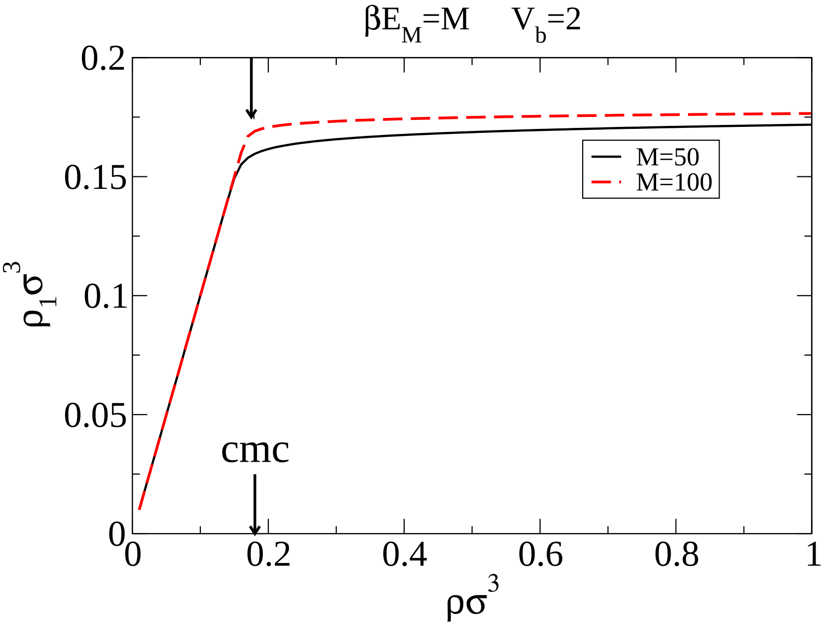

Fig. 5. – Monomer density ρ_1 as a function of the total density ρ for a system forming micelles composed each of $M = 100$ particles. The critical micelle concentration (cmc) marks the cross-over from the monomeric state to the aggregated micellar state. Beyond the cmc, increasing the density results in an increase of the number of micelles. No significant changes of the monomer density takes place.

9. – How do we "exactly" calculate Q_n

Up to now we have made use of simple models, like the square-well or the Kern-Frenkel potentials, for which analytic evaluation of the partition functions can be provided under reasonable assumptions (see for example Wertheim theory of associations [18, 19]). Here we discuss how the cluster partition functions can be evaluated numerically[20, 21, 22]. An efficient numerical method is outlined in refs. [22, 23]. With this method, the relations between the various $\mathcal{Q}_n$ is obtained directly from a grand-canonical Monte Carlo (GCMC) simulation, i.e. a simulation at fixed T, V and chemical potential μ. The simulation starts with a single cluster and rejects all moves (insertion, deletion, translation or rotation) as a result of which the system would contain more than one cluster. By imposing the constraint of simulating only a single cluster, in the grand-canonical ensemble, the probability $\mathcal{P}(n)$ of observing a cluster of size n is

$$\mathcal{P}(n) = \frac{\exp{(n\beta\mu)}Q_n}{\sum_n \exp{(n\beta\mu)}Q_n} \tag{57}$$

so that

$$\frac{\mathcal{P}(n)}{\mathcal{P}(1)} = \frac{\mathcal{Q}_n}{\mathcal{Q}_1}\exp{[\beta\mu(n-1)]}. \tag{58}$$

Hence, the ratio $\mathcal{Q}_n/\mathcal{Q}_1$ can be directly obtained for all n from a GCMC simulation. Note that $\mathcal{Q}_n/\mathcal{Q}_1$ is independent of μ, and therefore one can set $\mu = 0$ in the grand-canonical simulation without loss of generality. With this choice,

$$\mathcal{Q}_n = \frac{\mathcal{P}(n)}{\mathcal{P}(1)}\mathcal{Q}_1 = \frac{V}{\Lambda^3}\frac{\mathcal{P}(n)}{\mathcal{P}(1)}. \tag{59}$$

The procedure can be numerically optimised in several ways, as described in refs. [22, 23]. It can also be generalized to evaluate partition functions of clusters with the same number of particles but with different topological properties (e.g. chains, rings, branched clusters). As a test of the method, fig. (6) shows a comparison between the cluster size distribution calculated with the described methodology and the cluster size observed in standard Monte Carlo simulations for a Kern-Frenkel Janus colloid model.

10. – Conclusions

In this short lecture, we have presented some basic self-assembly introductory concepts. Building on these concepts, We hope it will be possible to better grasp the contents of the following lectures, and the sophisticated level of understanding and exploitation of self-assembly which is today possible in colloidal science. Specifically, we like to recall two important aspects which we have not discussed but which are addressed in other Chapters of this book: (i) how to predict the structure of the aggregate from the knowledge of the interaction potential (direct) or how to design the interaction potential to spontaneously assemble a desired structure (inverse); (ii) how to exploit external fields to modify the pathways leading to self-assembly.

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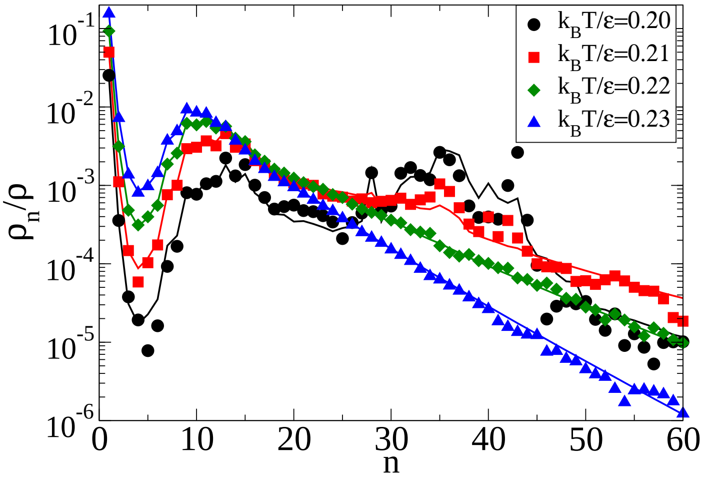

Fig. 6. – Cluster size distributions at a density $\rho\sigma^3 = 0.01$ as a function of cluster size n for four different temperatures $k_BT/ = 0.20$, $k_BT = 0.21$, $k_BT = 0.22$, and $k_BT = 0.23$. The lines indicate the results from the single-cluster simulations, and the points are directly obtained from NVT-simulations. Redrown from Ref. [23].

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