Calcoliamo la funzione di partizione nel canonico di singola particella

$$Z_1 = \frac{1}{h^2}\int dp_x dp_y \int dxdye^{-\beta(p^2/2m + Ar^4)} = \frac{1}{h^2}\frac{2\pi m}{\beta}\pi \int_0^\infty dr^2 e^{-\beta A r^4}$$

and changing from $\sqrt{\beta A} r^2$ to y

$$= \frac{\pi}{h^2}\frac{2\pi m}{\beta}\frac{1}{\sqrt{\beta A}}\int_0^\infty dye^{-y^2}$$

$$= \frac{1}{2}\frac{\pi}{h^2}\frac{2\pi m}{\beta}\frac{1}{\sqrt{\beta A}}\int_{-\infty}^\infty dye^{-y^2} = \frac{1}{2}\frac{\pi}{h^2}\frac{2\pi m}{\beta}\frac{1}{\sqrt{\beta A}}\sqrt{\pi} = \frac{m\pi^{5/2}}{h^2\beta^{3/2}\sqrt{A}}$$

Calcolo di $< E >$

$$< E_1 > = -\frac{\partial}{\partial\beta}\ln Z_1 = -\frac{\partial}{\partial\beta}\ln \beta^{-3/2} = \frac{3}{2}k_B T$$

Calcolo di $S(E)$

$$\beta F = -\ln Z = -N \ln Z_1 + \ln N!$$

$$\beta F = \beta(< E > - TS) = \frac{3}{2}N - \frac{S}{k_B}$$

$$\frac{S}{k_B} = \frac{3}{2}N + N\ln Z_1 - N\ln N + N = \frac{5}{2}N\ln e + N\ln\frac{m\pi^{5/2}}{h^2\beta^{3/2}\sqrt{A}} - N\ln N$$

$$= N\ln e^{5/2}\frac{m\pi^{5/2}}{Nh^2\beta^{3/2}\sqrt{A}}$$

e sostituendo

$$\beta = \frac{3N}{2E}$$

$$\frac{S(E)}{k_B} = N\ln\left[e^{5/2}\frac{m\pi^{5/2}}{Nh^2\sqrt{A}}\left(\frac{3N}{2E}\right)^{3/2}\right]$$

Calcolo dei momenti di r. La $P(r)$ e' data da

$$P(r) = \frac{re^{-\beta A r^4}}{\int re^{-\beta A r^4}dr}$$

I momenti sono dunque

$$< r^n > = \frac{\int r^{n+1}e^{-\beta A r^4}dr}{\int re^{-\beta A r^4}dr}$$

Cambiando variabile $y = \beta A r^4$, $(r = (\frac{y}{\beta A})^{1/4})$

$$< r^n >= \frac{\int r^{n+1-3} e^{-y} dy}{\int r^{1-3} e^{-y} dy} = (\beta A)^{\frac{n}{4}} \frac{\int y^{(n-2)/4} e^{-y} dy}{\int y^{-1/2} e^{-y} dy} = (\beta A)^{\frac{n}{4}} \frac{\int y^{(n-2)/4} e^{-y} dy}{\Gamma(1/2)}$$

Quindi, con $\Gamma(1/2) = \sqrt{\pi}$

$$< r >= (\beta A)^{\frac{1}{4}} \frac{\int y^{1/4} e^{-y} dy}{\sqrt{\pi}} = (\beta A)^{\frac{1}{4}} \frac{\Gamma(3/4)}{\sqrt{\pi}}$$

e

$$< r^2 >= (\beta A)^{\frac{1}{2}} \frac{\int e^{-y} dy}{\sqrt{\pi}} = (\beta A)^{\frac{1}{2}} \frac{1}{\sqrt{\pi}}$$

Il valore piu' probabile si ottine massimizzando $P(r)$

$$\frac{d}{dr} r e^{-\beta A r^4} = 0 \quad \rightarrow \quad e^{-\beta A r^4} - r(4\beta A r^3) e^{-\beta A r^4} \quad \rightarrow \quad r^4 = \frac{1}{4\beta A}$$

La densita' degli stati classica di singola particella

$$g(\epsilon) = \frac{(2\pi)^2}{h^2} \int p dp \int r dr \delta \left(\frac{p^2}{2m} + A r^4 - \epsilon \right)$$

e con il cambio di variabile $y = p^2/2m$ e $t = A r^4$, $dy = \frac{p}{m} dp$ e $dt = 4 A r^3 dr$

$$g(\epsilon) = \frac{(2\pi)^2}{h^2} \frac{m}{\sqrt{2A}} \int dy \int t^{-1/2} dt \delta \left(y + t - \epsilon \right)$$

L' integrale su y diventa una θ

$$g(\epsilon) = \frac{(2\pi)^2}{h^2} \frac{m}{\sqrt{2A}} \int t^{-1/2} \theta \left(\epsilon - t \right) dt$$

e quindi $t < \epsilon$. Per cui

$$g(\epsilon) = \frac{(2\pi)^2}{h^2} \frac{m}{\sqrt{2A}} \theta(\epsilon) \int_0^{\epsilon} t^{-1/2} dt = \frac{(2\pi)^2}{h^2} \frac{m}{\sqrt{2A}} \theta(\epsilon) 2 \epsilon^{\frac{1}{2}} = \frac{(2\pi)^2}{h^2} \frac{\sqrt{2} m}{\sqrt{A}} \theta(\epsilon) \epsilon^{\frac{1}{2}}$$

Condensazione di Bose. Essendo l' energia minima $\epsilon = 0$, $\mu = 0$ e dunque

$$N = \frac{(2\pi)^2}{h^2} \frac{\sqrt{2} m}{\sqrt{A}} \int_0^{\infty} \frac{\epsilon^{\frac{1}{2}}}{e^{\beta \epsilon} - 1} d\epsilon = \frac{(2\pi)^2}{h^2} \frac{\sqrt{2} m}{\sqrt{A}} \frac{1}{\beta^{3/2}} \int_0^{\infty} \frac{y^{\frac{1}{2}}}{e^y - 1} dy$$

e chiamando $g(1) = \int_0^{\infty} \frac{y^{\frac{1}{2}}}{e^y - 1} dy$

$$N = \frac{(2\pi)^2}{h^2} \frac{\sqrt{2} m}{\sqrt{A}} \frac{1}{\beta^{3/2}} g(1)$$

da cui, visto che converge, si puo' stimare T_c.

La stessa espressione, uguagliata a $N/2$ a sinistra, da la risposta alla successiva domanda.

Per l' energia di Fermi dobbiamo calcolare

$$\int_0^{\epsilon_F} g(\epsilon)d\epsilon = N$$

$$N = \frac{(2\pi)^2}{h^2}\frac{\sqrt{2}m}{\sqrt{A}}\int_0^{\epsilon_F}\epsilon^{\frac{1}{2}}d\epsilon = \frac{2}{3}\frac{(2\pi)^2}{h^2}\frac{\sqrt{2}m}{\sqrt{A}}\epsilon_F^{3/2}$$

da cui invertendo di trova ϵ_F.

Per calcolare $< r >$ a $T = 0$ dobbiamo sommare su tutti gli stati con $\epsilon < \epsilon_F$.

Dunque

$$< r >= \frac{\frac{1}{h^2}\int dp_x dp_y \int r dx dy \left[1-\theta\left(\frac{p^2}{2m}+Ar^4-\epsilon_F\right)\right]}{\frac{1}{h^2}\int dp_x dp_y \int dx dy \left[1-\theta\left(\frac{p^2}{2m}+Ar^4-\epsilon_F\right)\right]}$$

quindi

$$0 < \frac{p^2}{2m} + Ar^4 < \epsilon_F \qquad \rightarrow \qquad \frac{p^2}{2m} < \epsilon_F - Ar^4$$

(con $r < (\epsilon_F/A)^{1/4}$)

$$< r >= \frac{\int_0^{2m(\epsilon_F-Ar^4)} dp^2 \int r^2 dr}{\int_0^{2m(\epsilon_F-Ar^4)} dp^2 \int r dr} = \frac{\int_0^{(\epsilon_F/A)^{1/4}} r^2 dr(\epsilon_F - Ar^4)}{\int_0^{(\epsilon_F/A)^{1/4}} r dr(\epsilon_F - Ar^4)}$$

Il numeratore da

$$\int_0^{(\epsilon_F/A)^{1/4}} r^2 dr(\epsilon_F - Ar^4) = \frac{1}{3}(\epsilon_F/A)^{3/4}\epsilon_F - A(\epsilon_F/A)^{7/4}\frac{1}{7} = A(\epsilon_F/A)^{7/4}\left(\frac{1}{3}-\frac{1}{7}\right)$$

Il denominatore da

$$\int_0^{(\epsilon_F/A)^{1/4}} r dr(\epsilon_F - Ar^4) = \frac{1}{2}(\epsilon_F/A)^{2/4}\epsilon_F - A(\epsilon_F/A)^{6/4}\frac{1}{6} = A(\epsilon_F/A)^{6/4}\left(\frac{1}{2}-\frac{1}{6}\right)$$

da cui

$$< r >= (\epsilon_F/A)^{1/4}\frac{4}{21}\frac{12}{4} = \frac{4}{7}(\epsilon_F/A)^{1/4}$$