6 Novembre 2018

Calcolo della funzione di partizione di singola particella

$$Z_1 = \frac{1}{h^2} \int dp_x dp_y e^{-\beta p^2/2m} \int dx dy e^{-\beta m\omega^2 (x+y)^2/2}$$

L' integrale sui momenti e' standard. Per integrare sulle coordinate occorre cambiare variabile ($t = x+y$ e $y = y$) con Jacobiano unitario. Nota anche che il campo di integrazione non cambia. Per cui

$$Z_1 = \frac{1}{h^2}\frac{2\pi m}{\beta} \int_{-\infty}^{\infty} dt \int_{-L/2}^{L/2} dy e^{-\beta m\omega^2 t^2/2} = \frac{1}{h^2}\frac{2\pi m}{\beta} L\sqrt{\frac{2\pi}{\beta m\omega^2}} = \frac{1}{h^2}\frac{(2\pi)^{3/2}\sqrt{m}}{\beta^{3/2}}\frac{L}{\omega}$$

L' energia

$$E_1 = -\frac{\partial}{\partial\beta}\ln Z_1 = -\frac{\partial}{\partial\beta}\ln\beta^{-3/2} = \frac{3}{2}k_B T$$

$$C_v = \frac{3}{2}k_B N$$

Per calcolare $S(E)$

$$\beta F = \beta < E > -\frac{S}{k_B T}$$

$$\frac{S}{k_B T} = \ln Z + \beta < E >= N\ln Z_1 - \ln N! + \beta < E >=$$

$$\frac{3}{2}N + N\ln\left(\frac{1}{h^2}\frac{2\pi m}{\beta}L\sqrt{\frac{2\pi}{\beta m\omega^2}}\right) - N\ln N + N$$

$$= \frac{5}{2}N\ln e + N\ln\left(\frac{1}{h^2}\frac{2\pi m}{\beta}L\sqrt{\frac{2\pi}{\beta m\omega^2}}\frac{1}{N}\right) = N\ln\left(e^{5/2}\frac{1}{h^2}\frac{2\pi m}{\beta}\frac{L}{N}\sqrt{\frac{2\pi}{\beta m\omega^2}}\right)$$

ed infine sostituendo $\beta = \frac{3N}{2E}$

$$= N\ln\left[e^{5/2}\frac{(2\pi)^{3/2}}{h^2}\sqrt{m}L\frac{1}{N\omega}\left(\frac{2E}{3N}\right)^{3/2}\right]$$

Per calcolare il potenziale chimico occorre calcolare la derivata dell' energia libera.

$$\beta\mu = \frac{\partial}{\partial N}\beta F = -\frac{\partial}{\partial N}\ln Z = -\frac{\partial}{\partial N}\ln\frac{Z_1^N}{N!} =$$

$$-\ln Z_1 + \frac{\partial}{\partial N}(N\ln N - N) = -\ln Z_1 + (\ln N + 1 - 1) = \ln N - \ln Z_1 = -\ln(Z_1/N)$$

$$= -\ln \frac{1}{h^2} \frac{(2\pi)^{3/2}\sqrt{m}}{\beta^{3/2}} \frac{L}{N\omega}$$

Per calcolare la densitá degli stati

$$g(\epsilon) = \frac{1}{h^2} \int dp_x dp_y \int dxdy \; \delta \left(\frac{p^2}{2m} + \frac{1}{2} m\omega^2 (x+y)^2 - \epsilon \right)$$

e con il solito cambio di variabile $z = p^2/2m$ e $mdz = pdp$

$$= \frac{2\pi}{h^2} \int pdp \int dtdy \; \delta \left(\frac{p^2}{2m} + \frac{1}{2} m\omega^2 t^2 - \epsilon \right) = \frac{2\pi}{h^2} m \int dz \int dtdy \; \delta \left(z + \frac{1}{2} m\omega^2 t^2 - \epsilon \right) =$$

$$\frac{2\pi}{h^2} mL \int_{-\infty}^{\infty} dt \; \theta(\epsilon - \frac{1}{2} m\omega^2 t^2)$$

e con l' ulteriore cambio di variabile $q = \frac{1}{2} m\omega^2 t^2$, $dt = q^{-1/2} dq / \sqrt{2m\omega^2}$

$$= \frac{2\pi}{h^2} mL \frac{1}{\sqrt{2m\omega^2}} 2 \int_0^{\infty} q^{-1/2} dq \theta(\epsilon - q)$$

Quindi poiche' $\epsilon > q$, si riduce a

$$g(\epsilon) = \frac{2\pi}{h^2} mL \frac{1}{\sqrt{2m\omega^2}} 2\theta(\epsilon) \int_0^{\epsilon} q^{-1/2} dq = \frac{8\pi}{h^2} mL \frac{1}{\sqrt{2m\omega^2}} \epsilon^{1/2} \; \theta(\epsilon)$$

Possiamo adesso calcolare la $P(\epsilon)$

$$P(\epsilon) = \frac{g(\epsilon) e^{-\beta\epsilon}}{Z_1} = \frac{\frac{8\pi}{h^2} mL \frac{1}{\sqrt{2m\omega^2}} \epsilon^{1/2} \; \theta(\epsilon) e^{-\beta\epsilon}}{\frac{1}{h^2} \frac{(2\pi)^{3/2}\sqrt{m}}{\beta^{3/2}} \frac{L}{\omega}} = \frac{2}{\sqrt{\pi}} \beta^{3/2} \epsilon^{1/2} e^{-\beta\epsilon} \; \theta(\epsilon)$$

Ora $< \epsilon >$ (anche se sappiamo che sara' $< E_1 >$)

$$< \epsilon > = \int_0^{\infty} \epsilon P(\epsilon) d\epsilon = \frac{2}{\sqrt{\pi}\beta} \int (\beta\epsilon)^{3/2} e^{-\beta\epsilon} d\beta\epsilon = \frac{2}{\sqrt{\pi}\beta} \Gamma[5/2]$$

e poiche' $\Gamma[5/2] = \frac{3}{4}\sqrt{\pi}$

$$< \epsilon > = \frac{3}{2} k_B T$$

Per calcolare $< \epsilon^2 >$

$$< \epsilon^2 > = \int_0^{\infty} \epsilon^2 P(\epsilon) d\epsilon = \frac{2}{\sqrt{\pi}\beta^2} \int (\beta\epsilon)^{5/2} e^{-\beta\epsilon} d\beta\epsilon = \frac{2}{\sqrt{\pi}\beta^2} \Gamma[7/2]$$

e poiche' $\Gamma[7/2] = \frac{15}{8}\sqrt{\pi}$

$$< \epsilon^2 > = \frac{15}{4} (k_B T)^2$$

La varianza e' dunque

$$\sigma_\epsilon^2 =< \epsilon^2 > - < \epsilon >^2 = \left(\frac{15}{4} - \frac{9}{4}\right)(k_B T)^2 = \frac{3}{2}(k_B T)^2$$

Per trovare T_0 (BE) occorre calcolare

$$\tilde{N} = \int g(\epsilon) n_{BE}(\epsilon) d\epsilon = \frac{8\pi}{h^2} mL \frac{1}{\sqrt{2m\omega^2}} \int \epsilon^{1/2}\ \theta(\epsilon) \frac{1}{e^{\beta\epsilon - \mu} - 1} d\epsilon$$

e poiché a $T = 0$ $\mu = \epsilon_{min} = 0$

$$\tilde{N} = \int g(\epsilon) n_{BE}(\epsilon) d\epsilon = \frac{8\pi}{h^2} mL \frac{1}{\sqrt{2m\omega^2}} \int_0^\infty \epsilon^{1/2} \frac{1}{e^{\beta\epsilon} - 1} d\epsilon$$

che converge per $\epsilon \to 0$. Esiste quindi la transizione di BE. T_0 si ottiene da

$$N = \frac{8\pi}{h^2} mL \frac{1}{\sqrt{2m\omega^2}} \beta_0^{-3/2} \frac{\sqrt{\pi}}{2} \zeta(3/2)$$

dove abbiamo identificato la funzione $\frac{\sqrt{\pi}}{2}\zeta(3/2) = \int_0^\infty (\beta\epsilon)^{1/2} \frac{1}{e^{\beta\epsilon}-1} d\beta\epsilon$.

Per trovare il numero di particelle nel condensato a $T = T_0/2$ notiamo che per $T < T_0$ il prodotto

$$N_{livelli} \beta^{3/2} = \text{costante}$$

per cui (indicando con $N_{livelli}(T_0/2) = N$)

$$N_{livelli}(T_0) \beta_0^{3/2} = N_{livelli}(T_0/2)(2\beta_0)^{3/2}$$

per cui

$$N_{livelli}(T_0/2) = \left(\frac{1}{2}\right)^{3/2} N$$

e

$$N_{condensato} = N - N_{livelli}(T) \quad \to \quad N_{condensato}(T_0/2) = N - \left(\frac{1}{2}\right)^{3/2} N$$

Infine per simmetria possiamo concludere che $< y >= 0$.