Iniziamo con il calcolare la funzione di partizione di singola particella classica

$$Q_1 = \frac{1}{h^3}\int d\vec{p}d\vec{r}e^{-\beta p^2/2m-\beta\alpha r^3} = \frac{1}{h^3}\left(\frac{2\pi m}{\beta}\right)^{3/2} 4\pi \int r^2 dr e^{-\beta\alpha r^3}$$

e definendo $t = \alpha\beta r^3$, $dt = 3\alpha\beta r^2 dr$,

$$= \left(\frac{8\pi^2 m}{h^2\beta}\right)^{3/2} \frac{1}{3\alpha\beta}\int_0^{\alpha\beta R^3} dt e^{-t} = \left(\frac{8\pi^2 m}{h^2\beta}\right)^{3/2} \frac{1}{3\alpha\beta}\left(1-e^{-\alpha\beta R^3}\right)$$

Calcoliamo $< E_1 >$

$$< E_1 > = -\frac{\partial}{\partial\beta}\ln Q_1 = -\frac{\partial}{\partial\beta}\left(\ln\beta^{-5/2} + \ln\left(1-e^{-\alpha\beta R^3}\right)\right) =$$

$$\frac{5}{2}k_B T - \frac{\alpha R^3 e^{-\alpha\beta R^3}}{(1-e^{-\alpha\beta R^3})} = \frac{5}{2}k_B T - \frac{\alpha R^3}{e^{\alpha\beta R^3}-1}$$

Da qui, derivando, possiamo calcolare il calore specifico a volume costante

$$C_V = N\frac{d}{dT} < E_1 > = N\frac{5}{2}k_B + N\alpha R^3 \frac{1}{k_B T^2}\frac{d}{d\beta}\frac{1}{(e^{\alpha\beta R^3}-1)}$$

$$= N\frac{5}{2}k_B + N\alpha R^3\frac{1}{k_B T^2}\frac{\alpha R^3 e^{\alpha\beta R^3}}{(e^{\alpha\beta R^3}-1)^2} = N\frac{5}{2}k_B + \frac{N}{k_B T^2}\frac{(\alpha R^3)^2 e^{\alpha\beta R^3}}{(e^{\alpha\beta R^3}-1)^2}$$

Per calcolare la pressione, effettuiamo una variazione isotropica del volume

$$\beta P = -\frac{\partial\beta F}{\partial V} = \frac{\partial\ln Q_N}{\partial V} = N\frac{\partial\ln Q_1}{\partial V} = N\frac{\partial\ln\left(1-e^{-\alpha\beta R^3}\right)}{\frac{4\pi}{3}\partial R^3} = \frac{3N}{4\pi}\frac{\alpha\beta e^{-\alpha\beta R^3}}{1-e^{-\alpha\beta R^3}}$$

da cui

$$P(R) = \frac{3N}{4\pi}\frac{\alpha}{e^{\alpha\beta R^3}-1}$$

Per calcolare $< r >$, occorre definire $P(r)$ (dove P ora indica la densita' di probabilita') tale che

$$\int P(r)dr = 1 \qquad \rightarrow \qquad P(r)dr = \frac{r^2 e^{-\beta\alpha r^3}dr}{\int_0^R r^2 e^{-\beta\alpha r^3}dr}$$

per cui, cambiando variabile $y = \beta\alpha r^3$, $dy = 3\beta\alpha r^2 dr$

$$< r > = \int r P(r) dr = \frac{\int_0^R r^3 e^{-\beta\alpha r^3}dr}{\int_0^R r^2 e^{-\beta\alpha r^3}dr} = \frac{\int_0^{\beta\alpha R^3}\left(\frac{y}{\beta\alpha}\right)^{1/3} e^{-y}dy}{\int_0^{\beta\alpha R^3} e^{-y}dy} =$$

$$\frac{\int_0^{\beta\alpha R^3}\left(\frac{y}{\beta\alpha}\right)^{1/3} e^{-y}dy}{1-e^{-\beta\alpha R^3}}$$

Per basse temperature, $\beta \to \infty$,

$$<r>=\left(\frac{1}{\beta\alpha}\right)^{1/3}\Gamma[4/3]$$

La T a cui 3/4 delle particelle ha $r < R/2^{1/3}$ si calcola come

$$\int_0^{R/2^{1/3}} P(r)dr = \frac{3}{4}$$

$$\frac{\int_0^{R/2^{1/3}} r^2 e^{-\beta\alpha r^3}dr}{\int_0^R r^2 e^{-\beta\alpha r^3}dr} = \frac{3}{4}$$

e con il solito cambio di variabile $y = \beta\alpha r^3$,

$$\frac{\int_0^{R/2^{1/3}} e^{-y}dy}{\int_0^R e^{-y}dy} = \frac{3}{4} = \frac{1-e^{-\beta\alpha R^3/2}}{1-e^{-\beta\alpha R^3}}$$

da cui, chiamando $z = e^{-\beta\alpha R^3/2}$

$$3-3z^2 = 4-4z \quad \to 3z^2-4z+1=0$$

$$z = \frac{4\pm\sqrt{16-12}}{6} \to z_1 = 1; z_2 = \frac{1}{3}$$

cosi che'

$$-\beta\alpha R^3/2 = \ln\frac{1}{3} \to k_B T = \frac{\alpha R^3}{2\ln 3}$$

Calcoliamo adesso la densita' degli stati "classica"

$$g(\epsilon) = \frac{1}{h^3}\int d\vec{p}d\vec{r}\delta(p^2/2m+\alpha r^3-\epsilon)$$

e cambiando variabile $y = p^2/2m$, $mdy = pdp$ e $t = \alpha r^3$, $dt = 3\alpha r^2 dr$

$$g(\epsilon) = \frac{1}{h^3}4\pi\sqrt{2}(m)^{3/2}\frac{4\pi}{3\alpha}\int dt\int y^{0.5}dy\delta(y+t-\epsilon)$$

ed integrando su y

$$g(\epsilon) = \frac{1}{h^3}4\pi\sqrt{2}(m)^{3/2}\frac{4\pi}{3\alpha}\int_0^{\alpha R^3} dt\theta(\epsilon-t)\sqrt{\epsilon-t}$$

e definendo $z = \epsilon - t$

$$= \frac{1}{h^3} 4\pi\sqrt{2}(m)^{3/2} \frac{4\pi}{3\alpha} \int_{-\alpha R^3+\epsilon}^{\epsilon} dz \theta(z)\sqrt{z}$$

Poiche' l' integrale va da $\epsilon - \alpha R^3$ a ϵ, $z = 0$ (l' inizio della θ) puo' essere in 3 diversi casi

$$\begin{cases} 0 < \epsilon - \alpha R^3 & \int_{\epsilon-\alpha R^3}^{\epsilon} \\ \epsilon - \alpha R^3 < 0 < \epsilon & \int_0^{\epsilon} \\ 0 > \epsilon & \int = 0 \end{cases}$$

queste 3 condizioni diventano

$$g(\epsilon) = \frac{1}{h^3} 4\pi\sqrt{2}(m)^{3/2} \frac{4\pi}{3\alpha} \left\{ \theta(\epsilon - \alpha R^3) \frac{2}{3} [\epsilon^{3/2} - (\epsilon - \alpha R^3)^{3/2}] + (1 - \theta(\epsilon - \alpha R^3))\theta(\epsilon) \frac{2}{3} (\epsilon)^{3/2} \right\}$$

Il prodotto $\theta(\epsilon)\theta(\epsilon - \alpha R^3)$, visto come funzione di ϵ e' $\theta(\epsilon - \alpha R^3)$. Quindi

$$g(\epsilon) = \frac{2}{3} \frac{1}{h^3} 4\pi\sqrt{2}(m)^{3/2} \frac{4\pi}{3\alpha} \left\{ \theta(\epsilon - \alpha R^3)[\epsilon^{3/2} - (\epsilon - \alpha R^3)^{3/2}] + \theta(\epsilon) \frac{2}{3} (\epsilon)^{3/2} - \theta(\epsilon - \alpha R^3))(\epsilon)^{3/2} \right\}$$

$$g(\epsilon) = \frac{(2m)^{3/2}(4\pi)^2}{9\alpha h^3} \left\{ \theta(\epsilon)(\epsilon)^{3/2} - \theta(\epsilon - \alpha R^3)(\epsilon - \alpha R^3)^{3/2}] \right\}$$

Per capire se esiste condensazione di BE occorre guardare la seguente equazione (visto che $\mu = 0$)

$$N = \int d\epsilon g(\epsilon) \frac{1}{e^{\beta\epsilon} - 1}$$

$$N = \frac{(2m)^{3/2}(4\pi)^2}{9\alpha h^3} \left\{ \int_0^{\infty} \frac{\epsilon^{3/2}}{e^{\beta\epsilon} - 1} d\epsilon - \int_{\alpha R^3}^{\infty} \frac{(\epsilon - \alpha R^3)^{3/2}}{e^{\beta\epsilon} - 1} d\epsilon \right\}$$

Nel limite di $\beta\epsilon \to 0$, l'integrale converge. Infatti, per a piccolo

$$\int_0^a \frac{\epsilon^{3/2}}{e^{\beta\epsilon} - 1} d\epsilon \approx \int_0^a \frac{\epsilon^{3/2}}{\beta\epsilon} d\epsilon = \frac{1}{\beta} \int_0^a \epsilon^{1/2} d\epsilon = \frac{2}{3} \frac{1}{\beta} a^{3/2}$$

Quindi esiste la condensazione di BE.

In piu' ricordiamo che

$$\int_0^{\infty} \frac{y^{3/2}}{e^y - 1} dy = \frac{3\sqrt{\pi}}{4} \zeta(5/2)$$

e che il secondo integrale si puo' cambiare ($t = \epsilon - \alpha R^3$) e poi $z = \beta t$. Dunque la temperatura T_0 e' soluzione di

$$N = \frac{(2m)^{3/2}(4\pi)^2}{9\alpha h^3} \beta_0^{-5/2} \left\{ \frac{3\sqrt{\pi}}{4} \zeta(5/2) - \int_0^{\infty} \frac{z^{3/2}}{e^{\beta_0 \alpha R^3} e^z - 1} dz \right\}$$

Nel limite $T << 1$, poiche' l' energia del condensato e' zero

$$< E >= \int d\epsilon g(\epsilon) \frac{\epsilon}{e^{\beta\epsilon} - 1}$$

$$< E >= \frac{(2m)^{3/2}(4\pi)^2}{9\alpha h^3} \left\{ \int_0^\infty \frac{\epsilon^{5/2}}{e^{\beta\epsilon} - 1} d\epsilon - \int_{\alpha R^3}^\infty \frac{(\epsilon - \alpha R^3)^{3/2}\epsilon}{e^{\beta\epsilon} - 1} d\epsilon \right\}$$

A bassa T verranno popolati preferenzialmente gli stati con piccola energia, e dunque si avranno contributi solo dal primo integrale

$$< E > \approx \frac{(2m)^{3/2}(4\pi)^2}{9\alpha h^3} \left\{ \int_0^\infty \frac{\epsilon^{5/2}}{e^{\beta\epsilon} - 1} d\epsilon \right\} = \frac{(2m)^{3/2}(4\pi)^2}{9\alpha h^3} \beta^{-\frac{7}{2}} \left\{ \int_0^\infty \frac{y^{5/2}}{e^y - 1} dy \right\}$$

e ricordando che

$$\int_0^\infty \frac{y^{5/2}}{e^y - 1} dy = \frac{15\sqrt{\pi}}{8} \zeta(7/2)$$

$$< E >= \frac{(2m)^{3/2}(4\pi)^2}{9\alpha h^3} \beta^{-\frac{7}{2}} \frac{15\sqrt{\pi}}{8} \zeta(7/2)$$

Passiamo ora a Fermioni con $g_s = 2$. Per calcolare l' energia di Fermi

$$N = g_s \int_0^{\epsilon_F} d\epsilon g(\epsilon) = g_s \frac{(2m)^{3/2}(4\pi)^2}{9\alpha h^3} \int_0^{\epsilon_F} d\epsilon \left\{ \theta(\epsilon)(\epsilon)^{3/2} - \theta(\epsilon - \alpha R^3)(\epsilon - \alpha R^3)^{3/2}] \right\}$$

$$= g_s \frac{(2m)^{3/2}(4\pi)^2}{9\alpha h^3} \left\{ \int_0^{\epsilon_F} d\epsilon(\epsilon)^{3/2} - \theta(\epsilon_F - \alpha R^3) \int_{\alpha R^3}^{\epsilon_F} (\epsilon - \alpha R^3)^{3/2} d\epsilon] \right\}$$

$$= g_s \frac{(2m)^{3/2}(4\pi)^2}{9\alpha h^3} \left\{ \frac{2}{5} \epsilon_F^{5/2} - \theta(\epsilon_F - \alpha R^3) \int_{\alpha R^3}^{\epsilon_F} (\epsilon - \alpha R^3)^{5/2} d\epsilon] \right\}$$

e la densita' si ottiene dividendo per $4\pi R^3/3$.

Nel caso (che ci servita' dopo) di $\epsilon_F = \alpha R^3$, il secondo integrale non contribuisce e

$$N(\epsilon_F = \alpha R^3) = g_s \frac{(2m)^{3/2}(4\pi)^2}{9\alpha h^3} \frac{2}{5} (\alpha R^3)^{5/2}$$

Per calcolare $< r^3 >$ calcoliamo $g(\epsilon, r)$ definita come

$$g(\epsilon, r) = \frac{1}{h^3} \int 4\pi p^2 dp \int 4\pi r'^2 dr' \delta(p^2/2m + \alpha r'^3 - \epsilon) \delta(r - r')$$

e con $y = p^2/2m$, $mdy = pdp$,

$$g(\epsilon, r) = \frac{8\pi^2}{h^3}(2m)^{\frac{3}{2}} \int y^{1/2} dy \int r'^2 dr' \delta(y + \alpha r'^3 - \epsilon)\delta(r - r') =$$

integrando su y

$$= \frac{8\pi^2}{h^3}(2m)^{\frac{3}{2}} \int_0^R (\epsilon - \alpha r'^3)^{1/2} r'^2 dr' \delta(r - r')\theta(\epsilon - \alpha r'^3) =$$

e inserendo anche la condizione $r < R$

$$= \frac{8\pi^2}{h^3}(2m)^{\frac{3}{2}} (\epsilon - \alpha r^3)^{\frac{1}{2}} r^2 \theta(\epsilon - \alpha r^3)\theta(R - r)$$

Calcoliamo adesso $< r^3 >$, quando $\epsilon_F = \alpha R^3$. Per comodita', chiamiamo

$$A \equiv \frac{8\pi^2}{h^3}(2m)^{\frac{3}{2}}$$

$$< r^3 >= \frac{g_s}{N(\epsilon_F)} \int_0^{\epsilon_F} d\epsilon \int_0^R r^3 g(\epsilon, r) dr$$

$$= \frac{g_s A}{N(\epsilon_F)} \int_0^{\epsilon_F} d\epsilon \int_0^R r^3 (\epsilon - \alpha r^3)^{\frac{1}{2}} r^2 \theta(\epsilon - \alpha r^3)\theta(R - r) dr$$

Con il cambio di variabile $\alpha r^3 = t$, $3\alpha r^2 dr = dt$, $tdt = 3\alpha^2 r^5 dr$

$$= \frac{g_s}{N(\epsilon_F)} A \frac{1}{3\alpha^2} \int_0^{\epsilon_F} d\epsilon \int_0^{\alpha R^3} tdt (\epsilon - t)^{\frac{1}{2}} \theta(\epsilon - t)$$

La $\theta(\epsilon - t)$ impone che l' integrale su ϵ parta da t. Dunque

$$< r^3 >= \frac{g_s}{N(\epsilon_F)} A \frac{1}{3\alpha^2} \int_0^{\alpha R^3} tdt \int_t^{\epsilon_F} d\epsilon (\epsilon - t)^{\frac{1}{2}} = \frac{g_s}{N(\epsilon_F)} A \frac{1}{3\alpha^2} \int_0^{\alpha R^3} tdt \int_0^{\epsilon_F - t} dy y^{\frac{1}{2}}$$

dove abbiamo chiamato $y = \epsilon - t$. Dunque

$$< r^3 >= \frac{g_s}{N(\epsilon_F)} A \frac{1}{3\alpha^2} \frac{2}{3} \int_0^{\alpha R^3} tdt (\alpha R^3 - t)^{\frac{3}{2}} = \frac{g_s}{N(\epsilon_F)} A \frac{1}{3\alpha^2} \frac{2}{3} \int_{\alpha R^3}^0 (\alpha R^3 - z)(-dz) z^{\frac{3}{2}}$$

dove abbiamo chiamato $z = \alpha R^3 - t$. Otteniamo cosi'

$$< r^3 >= \frac{g_s}{N(\epsilon_F)} A \frac{1}{3\alpha^2} \frac{2}{3} \left[\int_0^{\alpha R^3} \alpha R^3 z^{\frac{3}{2}} dz - \int_0^{\alpha R^3} z^{\frac{5}{2}} dz \right] = \frac{g_s}{N(\epsilon_F)} A \frac{1}{3\alpha^2} \frac{2}{3} \left[\alpha R^3 \frac{2}{5} (\alpha R^3)^{\frac{5}{2}} - \frac{2}{7} (\alpha R^3)^{\frac{7}{2}} \right]$$

$$= \frac{g_s}{N(\epsilon_F)} A \frac{1}{\alpha^2} \frac{2}{3} (\alpha R^3)^{\frac{7}{2}} \frac{14-10}{35} = \frac{g_s}{N(\epsilon_F)} \frac{8}{9 \cdot 35} \frac{1}{\alpha^2} \frac{8\pi^2}{h^3} (2m)^{\frac{3}{2}} (\alpha R^3)^{\frac{7}{2}}$$

Avevamo visto che

$$N(\epsilon_F = \alpha R^3) = g_s \frac{(2m)^{3/2}(4\pi)^2}{9\alpha h^3} \frac{2}{5} (\alpha R^3)^{5/2} \quad \frac{g_s}{N(\epsilon_F)} = \frac{9\alpha h^3}{(2m)^{3/2} 2 \cdot 8\pi^2} \frac{5}{2} (\alpha R^3)^{-5/2}$$

e semplificando i termini identici

$$< r^3 >= \frac{2}{7} R^3$$